

The Geometry of Stable Quotients in Genus One

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Abstract

Stable quotient spaces provide an alternative to stable maps for compactifying spaces of maps. When $n \geq 2$, the space $\overline{Q}_g(\mathbb{P}^{n-1}, d) = \overline{Q}_g(G(1, n), d)$ compactifies the space of degree d maps of smooth genus g curves to \mathbb{P}^{n-1} , while $\overline{Q}_g(G(1, 1), d) \simeq \overline{M}_{1, d-\epsilon}/S_d$ is a quotient of a Hassett weighted pointed space. In this paper we study the coarse moduli schemes associated to the smooth proper Deligne-Mumford stacks $\overline{Q}_1(\mathbb{P}^{n-1}, d)$, for all $n \geq 1$. We show these schemes are projective, rationally connected and have Picard number 2. Then we give generators for the Picard group, compute the canonical divisor, and the cones of ample and effective divisors. We conclude that $\overline{Q}_1(\mathbb{P}^{n-1}, d)$ is Fano if and only if $n(d-1)(d+2) < 20$. In the case $n = 1$, we write in addition a closed formula for the Poincaré polynomial.

1 Introduction

One of the foundational steps in the mathematical development of Gromov-Witten theory was Kontsevich's construction of moduli spaces of stable maps $\overline{M}_{g,m}(X, \beta)$ compactifying the space of maps of genus g curves with m marked points to a target X with image in a fixed homology class β . For the purposes of Gromov-Witten theory, the essential point is that although the moduli spaces are in general neither smooth nor of the expected dimension, they do carry a virtual class.

However, the geometry of the moduli spaces themselves is generally ill behaved. The simplest target to consider is \mathbb{P}^r . In this case, $\overline{M}_{0,m}(\mathbb{P}^r, d)$ is smooth (as a Deligne-Mumford stack), while for all other genera $g \geq 1$, the spaces $\overline{M}_{g,m}(\mathbb{P}^r, d)$ are generally singular and have multiple components of different dimensions. This behavior begins almost immediately - even $\overline{M}_{1,0}(\mathbb{P}^2, 3)$ compactifying the 9-dimensional classical moduli space of smooth plane cubics has 3 components, two of dimension 9 and one of dimension 10. So for $g \geq 1$, it has been difficult to study the geometry of the moduli spaces $\overline{M}_{g,m}(\mathbb{P}^r, d)$. In genus 0, Pandharipande gave generators for the Picard group, computed the canonical divisor, as well as intersections of divisors in $\overline{M}_{0,m}(\mathbb{P}^r, d)$ in [Pa97] and [Pa99].

In 2009, Marian-Oprea-Pandharipande defined stable quotient spaces, giving a new compactification of the space of maps of genus g curves to Grassmanians. These stable quotient spaces tend to be more efficient compactifications than stable maps. As with stable maps, for all genera the moduli spaces of stable quotients carry a virtual class. When the target is projective space, there is a map $c : \overline{M}_{g,m}(\mathbb{P}^{n-1}, d) \twoheadrightarrow \overline{Q}_{g,m}(\mathbb{P}^{n-1}, d)$ and the strongest possible comparison holds, that is

$c_*[\overline{M}_{g,m}(\mathbb{P}^{n-1}, d)]^{vir} = [\overline{Q}_{g,m}(\mathbb{P}^{n-1}, d)]^{vir}$ [MOP09]. For targets other than \mathbb{P}^{n-1} , such a simple relationship is not expected.

One approach to probing the relationship between stable maps and stable quotients was proposed by Toda in [T10]. There, he introduces a sequence of moduli spaces $\overline{Q}_{g,m}(G(r, n), d)^\epsilon$ $0 \leq \epsilon \in \mathbb{R}$ interpolating between $\overline{M}_{g,m}(G(r, n), d)$ and $\overline{Q}_{g,m}(G(r, n), d) = \overline{Q}_{g,m}(G(r, n), d)^0$. For fixed g, m, r, n, d , there are finitely many spaces $\overline{Q}_{g,m}(G(r, n), d)^\epsilon$ in this sequence, and the change of moduli spaces as the stability parameter ϵ varies is an example of wall crossing. When $r = 1$, whenever $\epsilon > \epsilon'$, there is a morphism $c^{\epsilon, \epsilon'} : \overline{Q}_{g,m}(G(r, n), d)^\epsilon \rightarrow \overline{Q}_{g,m}(G(r, n), d)^{\epsilon'}$. The composition of all these maps is equal to the map c above, and we have a finite sequence of spaces

$$\overline{M}_{g,m}(\mathbb{P}^{n-1}, d) \rightarrow \overline{Q}_{g,m}(\mathbb{P}^{n-1}, d)^{\epsilon_1} \rightarrow \dots \rightarrow \overline{Q}_{g,m}(\mathbb{P}^{n-1}, d).$$

Using these spaces it is possible to define a sequence of Gromov-Witten like invariants for a target variety $X \subset \mathbb{P}^{n-1}$. It is hoped that wall crossing formulae can be derived to understand how these invariants change as ϵ changes.

One feature of stable quotients is that in genus 1 when the target is projective space, the moduli spaces $\overline{Q}_1(\mathbb{P}^{n-1}, d)$ compactifying spaces of maps without marked points are smooth, and we can study their geometry directly. This in turn may help to compute genus 1 stable quotient invariants (albeit invariants that don't require markings), and as discussed above, in turn may help to compute genus 1 Gromov-Witten invariants. Previously, there have been other constructions of smooth compactifications of genus 1 maps. In [VZ06], Vakil and Zinger construct a smooth compactification $\widetilde{M}_{1,m}(\mathbb{P}^{n-1}, d)$ by sequentially blowing up loci in the Kontsevich stable maps moduli space. Zinger subsequently used these spaces to compute the genus 1 Gromov-Witten invariants for the quintic threefold, validating physicists' predictions. The relation between the Vakil-Zinger compactification and stable quotients can be thought of in the following way. We have maps

$$\widetilde{M}_{1,m}(\mathbb{P}^{n-1}, d) \rightarrow \overline{M}_{1,m}^0(\mathbb{P}^{n-1}, d) \rightarrow \overline{Q}_{1,m}(\mathbb{P}^{n-1}, d)$$

Where $\overline{M}_{1,m}^0(\mathbb{P}^{n-1}, d)$ denotes the main component of the moduli space of stable maps. The Vakil-Zinger space is in this sense larger than stable maps, while stable quotients are smaller, and in the case $m = 0$ here, smooth. The Vakil-Zinger spaces have the advantage of being smooth even with marked points.

Another feature of stable quotients is that they are easier to work with by localization than stable maps, for the simple reason that they have fewer \mathbb{C}^* fixed loci relative to the stable maps moduli spaces. This makes localization computations more feasible on these spaces, and this was used by Pandharipande-Pixton in [PP11] to compute relations in the tautological ring of M_g , in particular proving that the conjectured Faber-Zagier relations in fact hold in $R^*(M_g)$.

Another similar application is the use by Pandharipande of stable quotient spaces to obtain relations in the κ ring of the moduli of curves of compact type, $\kappa^*(M_{g,n}^c)$ [Pa09a], [Pa09b]. Doing so, he is

able to give generators and relations for these rings, as well as formulas for the Betti numbers.

Our goal in this paper is to understand the basic features of the geometry of the genus 1 spaces $\overline{\mathcal{Q}}_1(\mathbb{P}^{n-1}, d)$. $\overline{\mathcal{Q}}_1(\mathbb{P}^{n-1}, d)$ is a smooth Deligne-Mumford stack of dimension nd [MOP09]. Knowing this, natural questions to address are, what topological information can be computed? The cohomology ring? Betti numbers? Geometrically, we can ask for all the analogues to the known results in genus 0 about $\overline{M}_{0,n}(\mathbb{P}^{n-1}, d)$. What is the Picard group and canonical divisor? (Computed in genus 0 by Pandharipande in [Pa97] and [Pa99].) What are the cones of ample and effective divisors? (Computed in genus 0 by Coskun-Harris-Starr in [CHS08] and [CHS09]. Is it possible to run Mori's program on this moduli space? (Addressed in genus 0 by Chen-Chrissman-Coskun in [CCC09].)

We address most of these questions for the case of the moduli spaces $\overline{\mathcal{Q}}_1(\mathbb{P}^{n-1}, d)$. The main tools we use are the virtual Poincare polynomial, a Białynicki-Birula stratification when $n \geq 2$, and standard intersection theory.

The main results contained in this paper are the following:

First the Poincaré polynomial of $\overline{\mathcal{Q}}_1(G(1, 1), d)$ is computed, and the expected symmetry of Poincaré duality is seen directly in the proof.

Theorem 2.14 If d is odd,

$$P_{\overline{\mathcal{Q}}_1(G(1,1),d)}(t) = \frac{1}{2d} \left[d(1+t^2)(1+t^4)^{\frac{d-1}{2}} + \sum_{k|d} \phi\left(\frac{d}{k}\right)(1+t^{\frac{2d}{k}})^k \right] + \sum_{j=1}^{d-1} t^{2j}$$

and if d is even,

$$P_{\overline{\mathcal{Q}}_1(G(1,1),d)}(t) = \frac{1}{2d} \left[\frac{d}{2}(1+t^2)^2(1+t^4)^{\frac{d}{2}-2} + \frac{d}{2}(1+t^4)^{\frac{d}{2}} + \sum_{k|d} \phi\left(\frac{d}{k}\right)(1+t^{\frac{2d}{k}})^k \right] + \sum_{j=1}^{d-1} t^{2j}.$$

Using the Białynicki-Birula stratification for the moduli space, we are able to in certain cases compute the Poincare polynomial of $\overline{\mathcal{Q}}_1(\mathbb{P}^{n-1}, d)$, and generally, to show that:

Proposition 3.6 For all n, d , and all i odd, $h^i(\overline{\mathcal{Q}}_1(\mathbb{P}^{n-1}, d), \mathbb{Q}) = 0$. That is, the odd rational cohomology of $\overline{\mathcal{Q}}_1(\mathbb{P}^{n-1}, d)$ vanishes.

and that

Theorem 4.1 The Picard rank of $\overline{\mathcal{Q}}_1(\mathbb{P}^{n-1}, d)$ is equal to $h^2(\overline{\mathcal{Q}}_1(\mathbb{P}^{n-1}, d), \mathbb{Q}) = 2$.

The Picard rank being 2 greatly simplifies divisor computations on these moduli spaces, and we are able to describe much of the divisor theory of the moduli spaces $\overline{Q}_1(\mathbb{P}^{n-1}, d)$. The divisors D_j , D_{fd} , and D_b are defined in Section 4.1.

Theorem 4.3 The nef cone of $\overline{Q}_1(\mathbb{P}^{n-1}, d)$ is bounded by the divisors D_j and D_{fd} .

Corollary 4.4 $\overline{Q}_1(\mathbb{P}^{n-1}, d)$ is a projective scheme.

Theorem 4.5 The effective cone of $\overline{Q}_1(\mathbb{P}^{n-1}, d)$ is bounded by the divisors D_j and D_b .

Theorem 4.6 $K_{\overline{Q}_1(\mathbb{P}^{n-1}, d)} = \frac{(d-11+(d-1)(n-1))}{12} D_j - n D_b$.

Finally, by direct analysis, we prove that

Theorem 5.1 $\overline{Q}_1(\mathbb{P}^{n-1}, d)$ is rationally connected.

These results leave the following natural question unresolved - are Toda's ϵ -stable quotients related to running the minimal model program on the Kontsevich spaces $\overline{M}_{1,0}(\mathbb{P}^{n-1}, d)$? More precisely, if one were to run the minimal model program on the main component of $\overline{M}_{1,0}(\mathbb{P}^{n-1}, d)$, could the sequence of main components of Toda's spaces, culminating in $\overline{Q}_1(\mathbb{P}^{n-1}, d)$, arise as one of the possible outcomes? The main component of $\overline{M}_{1,0}(\mathbb{P}^{n-1}, d)$ is normal [HL08], [Z04], so it is possible to consider running the minimal model program on it. Being rationally connected by Theorem 5.1, the last space in such a sequence should be a Fano fiber space. So the first step in this line of questioning would be to check if $\overline{Q}_1(\mathbb{P}^{n-1}, d)$ is minimal in the sense that it admits a map to a smaller dimensional space Y such that all fibers are Fano.

At the moment, what we can say is that there is a projection $\pi : \overline{Q}_1(\mathbb{P}^{n-1}, d) \rightarrow \overline{M}_{1,1}$, and all but one fiber are known to be Fano. Specifically, the fiber over any smooth genus 1 curve is isomorphic to \mathbb{P}^{nd-1}/G for a finite group G , hence Fano. The remaining fiber, over the nodal \mathbb{P}^1 , is not normal, and it is not known if the normalization of that fiber is Fano or not.

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1.0.1 Definition of Stable Quotients

In this paper we work over \mathbb{C} , and by *curve* will mean a reduced connected scheme of pure dimension 1. We begin by recalling the definition of stable quotients, as introduced in [MOP09]. In [MOP09] a more general definition allowing marked points is given, but as we will be concerned only with the unmarked case, we state only that definition here.

Let C be a curve with at worst nodal singularities, of arithmetic genus g . A quotient q of the trivial sheaf

$$0 \rightarrow S \rightarrow \mathbb{C}^n \otimes \mathcal{O}_C \xrightarrow{q} Q \rightarrow 0$$

is a *quasi-stable quotient* if Q is locally free at the nodes of C . Quasi-stability implies that

- (i) the torsion subsheaf $\tau(Q) \subset Q$ has support contained in C^{ns} , the nonsingular locus of C , and
- (ii) S is a locally free sheaf on C .

Let k denote the rank of S . Given C and a quasi-stable quotient Q on C , the data (C, q) determines a *stable quotient* if the \mathbb{Q} -line bundle

$$\omega_C \otimes (\wedge^k S^*)^{\otimes \epsilon} \text{ is ample for all } 0 < \epsilon \in \mathbb{Q}. \quad (1)$$

No amount of positivity of $\wedge^k S^*$ can stabilize a genus 0 component unless it contains at least 2 nodes, and if such a component has exactly 2 nodes, then $\wedge^k S^*$ must have positive degree on it.

Two quasi-stable quotients

$$0 \rightarrow S \rightarrow \mathbb{C}^n \otimes \mathcal{O}_C \xrightarrow{q} Q \rightarrow 0 \quad \text{and} \quad 0 \rightarrow S' \rightarrow \mathbb{C}^n \otimes \mathcal{O}_C \xrightarrow{q'} Q' \rightarrow 0$$

are *strongly isomorphic* if $S, S' \subset \mathbb{C}^n \otimes \mathcal{O}_C$ are equal. An *isomorphism* of quasi-stable quotients

$$\phi : (C, q) \rightarrow (C', q')$$

is an isomorphism of curves $\phi : C \xrightarrow{\sim} C'$ such that the quotients q and $\phi^*(q')$ are strongly isomorphic.

In [MOP09], Marian-Oprea-Pandharipande prove the following:

Theorem 1.1 ([MOP09] Theorem 1). *The moduli space of stable quotients $\overline{Q}_g(G(k, n), d)$ parameterizing the data*

$$(C, 0 \rightarrow S \rightarrow \mathbb{C}^n \otimes \mathcal{O}_C \xrightarrow{q} Q \rightarrow 0)$$

with $\text{rank}(S) = k$ and $\deg(S) = -d$ is a separated and proper Deligne-Mumford stack of finite type over \mathbb{C} .

Theorem 1.2 ([MOP09] Proposition 1). *$\overline{Q}_1(G(1, n), d) = \overline{Q}_1(\mathbb{P}^{n-1}, d)$ is a smooth irreducible Deligne-Mumford stack of dimension nd for $d > 0$.*

1.1 Notation and conventions

The stability condition (1) constrains the isomorphism type of the underlying curve C . In genus 1, there are only two possibilities. If (C, q) is a stable quotient, either C is a smooth genus 1 curve, or C is isomorphic to C_i , a cycle of i rational curves, and S^* has positive degree on each component of C_i . Thus for a stable quotient in $\overline{\mathcal{Q}}_1(\mathbb{P}^{n-1}, d)$ the underlying curve C is either a smooth genus 1 curve or is isomorphic to C_i for $1 \leq i \leq d$. This gives a natural stratification on $\overline{\mathcal{Q}}_1(\mathbb{P}^{n-1}, d)$.

Following [MOP09],

Definition 1. Let $\overline{M}_{g,m|d}$ be the moduli space parameterizing genus g curves with markings $\{p_1, \dots, p_m\} \cup \{q_1, \dots, q_d\}$ $C^{ns} \subset C$ satisfying

(i) the points p_i are distinct,

(ii) the points q_j are distinct from the points p_i

with stability given by the ampleness of

$$\omega_C(\sum_{i=1}^m p_i + \epsilon \sum_{j=1}^d q_j)$$

for every strictly positive $\epsilon \in \mathbb{Q}$.

Note that the points q_j can collide. $\overline{M}_{g,m|d}$ is a special case of Hassett's more general construction of weighted pointed spaces of curves, namely where the points p_i each have weight 1 and q_j each have weight ϵ , for $\epsilon < \frac{1}{d}$.

We end with an alternate description of the moduli space $\overline{\mathcal{Q}}_1(G(1, n), d)$. Given a line bundle S and an inclusion $0 \rightarrow S \rightarrow \mathbb{C}^n \otimes \mathcal{O}_C$, the dual sequence is $\mathbb{C}^n \otimes \mathcal{O}_C \rightarrow S^* \rightarrow 0$ and is equivalent to the data of n sections $s_i \in H^0(S^*)$. So $\overline{\mathcal{Q}}_1(\mathbb{P}^{n-1}, d)$ can alternatively be described as parameterizing, up to isomorphism, tuples $(C, (s_0, \dots, s_{n-1}))$ where S^* is some line bundle of degree d and $s_i \in H^0(S^*)$, subject to the same stability condition (1). In this notation, $(C, (s_0, \dots, s_{n-1})) \sim (C', (s'_0, \dots, s'_{n-1}))$ if and only if there exists $\lambda \in \mathbb{C}^*$ and $\phi : C \xrightarrow{\sim} C'$ such that $\lambda \cdot s'_i \circ \phi = s_i$, for all i .

In the case $n = 1$ then, $\overline{\mathcal{Q}}_1(G(1, 1), d)$ parameterizes pairs (C, D) , C a curve of genus 1 and D an effective divisor of degree d on C , up to automorphisms of C . Stability requires D to have at least one point on any rational component with exactly two nodes, and $(C, D) \sim (C', D')$ if there is an isomorphism $\phi : C \rightarrow C'$ with $\phi(D) = D'$. We conclude that $\overline{\mathcal{Q}}_1(G(1, 1), d)$ is isomorphic to the quotient $\overline{M}_{1,0|d}/S_d$.

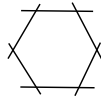


Figure 1: For example, the curve C_6 .

2 Topology of $\overline{Q}_1(G(1, 1), d)$

In this section we compute the Poincaré polynomial of $\overline{Q}_1(G(1, 1), d)$. The main tool used is the virtual Poincaré polynomial, which is additive on locally closed stratifications. The stratification we use here is the one introduced earlier, by isomorphism type of the underlying curve. $\overline{Q}_1(G(1, 1), d)$ has $d + 1$ strata: U_d , the strata where C is a smooth genus 1 curve, and Δ_d^i , the strata where $C \simeq C_i$, for $1 \leq i \leq d$. Our task reduces to computing the virtual Poincaré polynomial of each stratum. We begin by recording the definition and some basic properties of the virtual Poincaré polynomial.

2.1 Introduction to the Virtual Poincaré Polynomial

Given a compact Kähler manifold M and any $U \subset M$ the complement of a normal crossing divisor, there is a mixed Hodge structure on the cohomology groups with compact support $H_c^k(U, \mathbb{Q})$ [D71], [V03]. This mixed Hodge structure gives a weight filtration W on these cohomology groups, and the coefficients of the virtual Poincaré polynomial are defined from the dimensions of these graded pieces

Definition 2.

$$P_U^{vir}(t) := \sum_{m,k} (-1)^{k+m} \dim(Gr_W^m(H_c^k(U))) t^m.$$

We collect here a number of properties of the virtual Poincaré polynomial. The first two can be found in [F93].

- If X is a smooth complete orbifold, then the Poincaré polynomial and virtual Poincaré polynomial are equal, $P_X(t) = P_X^{vir}(t)$.
- If Z is a closed algebraic subset of X and $U = X \setminus Z$, then $P_X(t) = P_U(t) + P_Z(t)$. Hence $P_X^{vir}(t)$ is additive on locally closed strata.

We give proofs for the remaining three.

Lemma 2.1. *Given a fiber bundle $\pi : X \rightarrow B$ whose local systems $R^i\pi_*\mathbb{Q}$ are constant for all i , the virtual Poincaré polynomial is multiplicative, that is $P_X^{vir}(t) = P_B^{vir}(t) \times P_F^{vir}(t)$.*

Proof. The standard tool for computing the cohomology of the fiber bundle X is the Leray spectral sequence $E_r^{i,j} \Rightarrow H^{i+j}(X, \mathbb{Q})$. Deligne showed [D71] that for a projective fibration this spectral sequence degenerates at the E_2 term, where $E_2^{i,j} = H^i(B, R^j\pi_*\mathbb{Q})$. By M. Saito's theory of mixed Hodge modules, the Leray spectral sequence for cohomology with compact supports, which also degenerates at the E_2 term,

$$E_2^{i,j} = H_c^i(B, R^j\pi_*\mathbb{Q}) \Rightarrow H_c^{i+j}(X, \mathbb{Q})$$

is a spectral sequence of mixed Hodge structures [Sa89].

Assuming the local systems $R^i\pi_*\mathbb{Q}$ are trivial for all i , by an analogue of the universal coefficient theorem $H_c^i(B, R^j\pi_*\mathbb{Q}) = H_c^i(B, \mathbb{Q}) \otimes H_c^j(F, \mathbb{Q})$, so

$$\begin{aligned}
P_X^{vir}(t) &:= \sum_{m,k} (-1)^{k+m} \dim(Gr_W^m(H_c^k(X))) t^m \\
&= \sum_{m,k} (-1)^{k+m} \dim(Gr_W^m(H_c^k(B, R^j\pi_*\mathbb{Q}))) t^m \\
&= \sum_{a+b=m, i+j=k} (-1)^{i+j+a+b} \dim(Gr_W^a(H_c^i(B)) \otimes Gr_W^b(H_c^j(F))) t^m \\
&= \left(\sum_{a,i} (-1)^{a+i} \dim(Gr_W^a(H_c^i(B, \mathbb{Q}))) t^a \right) \left(\sum_{b,j} (-1)^{b+j} \dim(Gr_W^b(H_c^j(F, \mathbb{Q}))) t^b \right) \\
&= P_B^{vir}(t) \times P_F^{vir}(t).
\end{aligned}$$

□

Lemma 2.2. *If a finite group G acts holomorphically on \mathbb{P}^r , the virtual Poincaré polynomial of the orbifold \mathbb{P}^r/G is $1 + t^2 + \dots + t^{2r}$.*

Proof. If a finite group G acts holomorphically on a smooth projective variety X , $H^j(X/G, \mathbb{Q}) = H^j(X, \mathbb{Q})^G$. For smooth orbifolds the virtual Poincaré polynomial and Poincaré polynomial are equal so it will suffice to compute the ranks of $H^j(\mathbb{P}^r, \mathbb{Q})^G$.

For j odd, the rank is trivially zero. For j even, $H^j(\mathbb{P}^r, \mathbb{Q})$ is one-dimensional and generated by the class of a linear subspace S . Let

$$\alpha := \sum_{g \in G} g(S)$$

be the G -orbit of S . By assumption each $g \in G$ acts holomorphically on \mathbb{P}^r , in particular is orientation preserving. So there is no cancellation in this sum and α is nonzero and by construction in $H^j(\mathbb{P}^r, \mathbb{Q})^G$. We conclude that $H^j(\mathbb{P}^r, \mathbb{Q})^G$ is one-dimensional for j even. □

Lemma 2.3. *If a finite group G acts linearly on \mathbb{C}^r , the virtual Poincaré polynomial of the orbifold \mathbb{C}^r/G is t^{2r} .*

Proof. If G acts linearly on \mathbb{C}^r , this action can be extended to \mathbb{P}^r , and when this is done, the original \mathbb{C}^r and the \mathbb{P}^{r-1} at infinity are invariant subspaces of this action on \mathbb{P}^r . Hence

$$P^{vir}(\mathbb{P}^r/G) = P^{vir}(\mathbb{C}^r/G) + P^{vir}(\mathbb{P}^{r-1}/G),$$

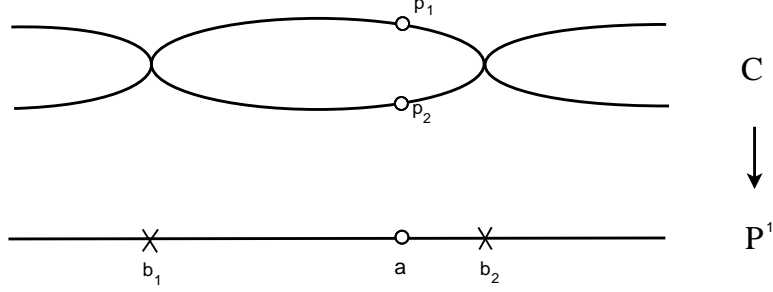


Figure 2: Map from U_2 to $M^{(4,1)}$.

the previous lemma applies to both projective spaces, and we have

$$1 + t^2 + \dots + t^{2r} = P^{vir}(\mathbb{C}^r/G) + 1 + t^2 + \dots + t^{2r-2},$$

completing the proof. \square

2.2 Computation of the virtual Poincaré polynomial of the main stratum U_d

2.2.1 The cases $d = 1$ and $d = 2$

For $d \geq 3$, the moduli space $M_{1,1}[d]$ of elliptic curves with full level d structure will be central to our computation of the virtual Poincaré polynomial of U_d . Because $M_{1,1}[d]$ is a fine moduli space only when $d \geq 3$, our general approach cannot be applied when $d = 1$ and $d = 2$. We begin by computing the virtual Poincaré polynomials of U_1 and U_2 directly.

When $d = 1$, $U_1 \simeq M_{1,1}$ so

$$P_{U_1}^{vir} = t^2. \quad (2)$$

Let $M^{(4,1)}$ denote the moduli space of \mathbb{P}^1 with five points $([b_1, b_2, b_3, b_4], a)$ on it, where the points b_i are unlabelled and not permitted to collide with each other, and there is no constraint on the point a . When $d = 2$, $U_2 \simeq M^{(4,1)}$ and the isomorphism Φ between these spaces is the following. Take a point $(C, (p_1 + p_2)) \in U_2$. The linear system $H^0(\mathcal{O}_C(p_1 + p_2))$ determines a double cover $f : C \rightarrow \mathbb{P}^1$. Let b_1, \dots, b_4 be the distinct branch points of f and $a = f(p_1) = f(p_2)$. Then $\Phi(C, (p_1 + p_2)) = ([b_1, b_2, b_3, b_4], a)$.

To compute $P_{U_2}^{vir}(t)$, we use the following lemma, which is a special case of Theorem 5.4 of [GP05].

Lemma 2.4. *Let G be a connected algebraic group and X a quasi-projective variety a G -action. If the action of G on X is almost-free, that is, the isotropy groups of the action are finite, then $P_X^{vir}(t) = P_G^{vir}(t)P_{X/G}^{vir}(t)$.*

Let $\mathfrak{M}^{(4)}$ denote the parameter space of 4 distinct unlabelled points on a *fixed* \mathbb{P}^1 , and similarly $\mathfrak{M}^{(4,1)}$ the space of 4 distinct unlabelled points and 1 unconstrained point on again, a *fixed* \mathbb{P}^1 . PGL_2 acts naturally on both these spaces. Note that $\mathfrak{M}^{(4)}/PGL_2 \simeq \mathbb{A}^1$, because 4 distinct branch points on \mathbb{P}^1 , up to $Aut(\mathbb{P}^1)$, determines a smooth $g = 1$ curve, and conversely. So

$$U_2 \simeq M^{(4,1)} \simeq \mathfrak{M}^{(4,1)}/PGL_2 \simeq (\mathbb{P}^1 \times \mathfrak{M}^{(4)})/PGL_2$$

.

Applying Lemma 2.4,

$$P_{U_2}^{vir}(t) = P_{(\mathbb{P}^1 \times \mathfrak{M}^{(4)})/PGL_2}^{vir}(t) = \frac{P_{\mathbb{P}^1}^{vir}(t) P_{\mathfrak{M}^{(4)}}^{vir}(t)}{P_{PGL_2}^{vir}(t)} = P_{\mathbb{P}^1}^{vir}(t) P_{\mathfrak{M}^{(4)}/PGL_2}^{vir}(t) = P_{\mathbb{P}^1}^{vir}(t) P_{\mathbb{A}^1}^{vir}(t) = (1+t^2)(t^2). \quad (3)$$

For the remainder of the discussion of U_d , we assume $d \geq 3$.

2.2.2 Construction of U_d as the quotient of a projective bundle

To compute the virtual Poincaré polynomial of U_d , we give an explicit construction of U_d . We begin with some background about moduli spaces of elliptic curves. For all but two elliptic curves, the automorphism group of the curve is $\mathbb{Z}/2$. The exceptions are ξ_4 and ξ_6 , with automorphism groups $\mathbb{Z}/4$ and $\mathbb{Z}/6$, respectively. We will write $\widetilde{M}_{1,1} = M_{1,1} \setminus \{\xi_4, \xi_6\}$, and generally for any moduli space X of genus 1 curves and additional data, will denote by \widetilde{X} the moduli space of a genus 1 curve other than ξ_4 or ξ_6 , with the same additional data.

Definition 3. Fix ζ_d a d^{th} root of unity. $M_{1,1}[d]$ is the moduli space of elliptic curves with full level d structure, and parameterizes the data of an elliptic curve (C, p) with a basis (e_1, e_2) of the d -torsion of (C, p) , such that the Weil pairing $\langle e_1, e_2 \rangle = \zeta_d$.

$G = Sp_2(\mathbb{Z}/d)$ acts on $M_{1,1}[d]$ in the following way: for $g \in G$, $g \cdot (C, p, (e_1, e_2)) = (C, p, (ge_1, ge_2))$. The subgroup $\langle -1, -1 \rangle \subset Sp_2(\mathbb{Z}/d)$ acts trivially, and in fact $M_{1,1}$ is the quotient $\rho : M_{1,1}[d] \rightarrow M_{1,1}[d]/(Sp_2(\mathbb{Z}/d)/(\mathbb{Z}/2)) \simeq M_{1,1}$. Moreover, $Sp_2(\mathbb{Z}/d)/(\mathbb{Z}/2)$ acts freely on $\widetilde{M}_{1,1}[d]$, so $\widetilde{\rho} : \widetilde{M}_{1,1}[d] \rightarrow \widetilde{M}_{1,1}$ is étale, and ρ is ramified exactly over ξ_4 and ξ_6 .

As we've taken $d \geq 3$, $M_{1,1}[d]$ is a fine moduli space and carries a universal curve $\pi : \mathcal{C} \rightarrow M_{1,1}[d]$ together with a section σ . By cohomology and base change $\pi_* \mathcal{O}_{\mathcal{C}}(d\sigma)$ is a vector bundle over $M_{1,1}[d]$, because for any smooth genus 1 curve C $H^1(\mathcal{O}_C(dp)) = H^0(\omega_C(-dp)) = H^0(\mathcal{O}_C(-dp)) = 0$, so $R^1 \pi_* \mathcal{O}_{\mathcal{C}}(d\sigma) = 0$, while all higher $R^i \pi_* \mathcal{O}_{\mathcal{C}}(d\sigma)$ vanish on any curve.

The projectivization $P_d := \mathbb{P}(\pi_* \mathcal{O}_{\mathcal{C}}(d\sigma))$ is a \mathbb{P}^{d-1} bundle over $M_{1,1}[d]$. This is a fine moduli space parameterizing the data $(C, p, (e_1, e_2), [s])$ where (C, p) is an elliptic curve, (e_1, e_2) a basis for the

d -torsion of that elliptic curve with fixed Weil pairing, $s \in H^0(\mathcal{O}_C(dp))$ a meromorphic function on C with at worst a pole of order d at p (because $\mathcal{O}_C(dp)$ comes with a canonical embedding in K^*), and $[s]$ the class of that function up to scaling by \mathbb{C}^* .

There are two natural finite group actions on P_d . The first is by $H := (\mathbb{Z}/d)^2$. To describe this action, note that on a smooth elliptic curve (C, p) , $H^0(\mathcal{O}_C(dq - dp)) = 0$ unless q is a d -torsion, in which case $\mathcal{O}_C(dq - dp) \simeq \mathcal{O}_C$ and $H^0(\mathcal{O}_C(dq - dp)) = \mathbb{C}$. Let τ_q denote the translation of C sending p to q .

Fix $(\alpha, \beta) \in H$. Let $q = p + \alpha e_1 + \beta e_2$ and $\tau_{(\alpha, \beta)} = \tau_q$. Pick a nonzero $f_{\alpha, \beta} \in H^0(\mathcal{O}_C(dp - dq)) = \mathbb{C}$. Define the action of H on $\mathbb{P}(\pi_* \mathcal{O}_C(d\sigma))$ by $(\alpha, \beta) \cdot (C, p, (e_1, e_2), [s]) = (C, p, (e_1, e_2), [\tau_{(\alpha, \beta)}^{-1} \circ (f_{\alpha, \beta} \cdot s)])$. Because $f_{\alpha, \beta}$ was unique up to scaling, this action is well defined and H acts fiberwise on the bundle P_d .

The second action is by $G = Sp_2(\mathbb{Z}/d)$. Given $g \in G$, $g \cdot (C, p, (e_1, e_2), [s]) = (C, p, (ge_1, ge_2), [s])$. Unlike with the action of G on $M_{1,1}[d]$, the subgroup $\langle -1, -1 \rangle$ does not act trivially on P_d . Specifically, at any point $x \in \widetilde{M}_{1,1}[d]$ the isotropy group of this action is $\mathbb{Z}/2$, while the isotropy groups at (ξ_4, p) and (ξ_6, p) are $\mathbb{Z}/4$ and $\mathbb{Z}/6$, respectively. These isotropy groups all act nontrivially on the fibers of P_d , so in the quotient of P_d by G , the fibers over ξ_4 and ξ_6 will be further quotients by $\mathbb{Z}/2$ and $\mathbb{Z}/3$, respectively. On \widetilde{P}_d , the action of G can be considered in two parts - a $\mathbb{Z}/2$ action that acts fiberwise and an equivariant, free $G/\mathbb{Z}/2$ action.

Recall that U_d can be described as the moduli space parameterizing (C, D) , C a smooth genus 1 curve and D an effective divisor of degree d , up to automorphisms of C . There is a forgetful map $\Phi : P_d \rightarrow U_d$ defined on the moduli functors, $\Phi(C, p, (e_1, e_2), [s]) = (C, |s|)$, where $|s|$ is the zero set of the section s of $\mathcal{O}_C(dp)$. This induces a map of the schemes $P_d \rightarrow U_d$.

Proposition 2.5. $U_d = P_d/H/G$.

Proof. By Lemma 2.6 Φ is surjective, and by Lemma 2.7 the preimage of every point in U_d is a $G \times H$ orbit. \square

Lemma 2.6. Φ is surjective.

Proof. Fix a point $(C, D) \in U_d$. To show that Φ is surjective it will suffice to produce a point $p \in C$ and a meromorphic function $f \in H^0(\mathcal{O}_C(dp))$ such that D is the divisor of zeroes of f as a section of the line bundle $H^0(\mathcal{O}_C(dp))$. Pick any point $z \in C$ to produce an elliptic curve (C, z) . Suppose $D = p_1 + \dots + p_d$. Using the group law on the elliptic curve, we can take the sum $q = p_1 + \dots + p_d$, and then pick a point p such that $dp = q$. Finally, pick any basis (e_1, e_2) of the d -torsion of (C, p) with $\langle e_1, e_2 \rangle = \zeta_d$. Then $\Phi(C, p, (e_1, e_2), [f]) = (C, D)$. \square

Lemma 2.7. The preimage of each point in U_d under Φ is a $G \times H$ orbit.

Proof. Fix a point $(C, D) \in U_d$, and suppose $(C, p, (e_1, e_2), [f]), (C, p', (e'_1, e'_2), [f'])$ are both in $\Phi^{-1}(C, D)$. $\mathcal{O}(D) = \mathcal{O}(dp) = \mathcal{O}(dp')$ so p' is a d -torsion point of the elliptic curve (C, p) . There is

a unique $h = (\alpha, \beta) \in (\mathbb{Z}/d)^2$ such that $p' = p + \alpha e_1 + \beta e_2$. There is a unique $g \in Sp_2(\mathbb{Z}/d)$ such that $(e'_1, e'_2) = (g\tau_{\alpha,\beta}e_1, g\tau_{\alpha,\beta}e_2)$. It remains to show that

$$g \cdot h(C, p, (e_1, e_2), [f]) := (C, \tau_{\alpha,\beta}p, (g\tau_{\alpha,\beta}e_1, g\tau_{\alpha,\beta}e_2), [\tau_{\alpha,\beta}^*f]) = (C, p', (e'_1, e'_2), [f']).$$

All equalities but $[\tau_{\alpha,\beta}^*f] = [f']$ are by construction, and the last is true because the isomorphism $H^0(\mathcal{O}(dp))$ to $H^0(\mathcal{O}(dp'))$ is exactly $\tau_{\alpha,\beta}^*$.

□

G acts on \widetilde{P}_d by $\mathbb{Z}/2$ on the fibers and the quotient $G/(\mathbb{Z}/2)$ acts equivariantly. So we can study $\widetilde{P}_d/H/G$ by first taking the quotient \widetilde{P}_d/\hat{H} , where $\hat{H} := H \times \mathbb{Z}/2$ acts fiberwise, and then taking the further quotient by $\hat{G} := G/(\mathbb{Z}/2)$.

$$\begin{array}{ccc} \widetilde{P}_d & & \\ \downarrow & & \\ \widetilde{P}_d/\hat{H} & \longrightarrow & \widetilde{P}_d/\hat{H}/\hat{G} \simeq \widetilde{U}_d \\ \downarrow & & \downarrow \\ \widetilde{M}_{1,1}[d] & \longrightarrow & \widetilde{M}_{1,1} \end{array}$$

In the following, we will call X a fiber bundle over base B if X is an algebraic variety mapping to B and the map $f : X \rightarrow B$ is locally trivial in the etale topology.

Proposition 2.8. $\widetilde{U}_d = \widetilde{P}_d/H/G$ is a fiber bundle over $\widetilde{M}_{1,1}$, with fibers \mathbb{P}^{d-1}/\hat{H} .

Proof. Since \widetilde{P}_d is the projectivization of a vector bundle over $\widetilde{M}_{1,1}[d]$, it is Zariski locally trivial, which implies etale locally trivial. By Lemma 2.9 \widetilde{P}_d/\hat{H} is a \mathbb{P}^{d-1}/\hat{H} bundle over $\widetilde{M}_{1,1}[d]$. Since \hat{G} acts freely and equivariantly on \widetilde{P}_d/\hat{H} over $\widetilde{M}_{1,1}[d]$, the bundle descends to a fiber bundle $\widetilde{P}_d/H/G$ over $\widetilde{M}_{1,1}$ with fibers \mathbb{P}^{d-1}/\hat{H} . □

Lemma 2.9. Given a fiber bundle $X \rightarrow B$ with fibers \mathbb{P}^n and an action of a finite group H on X such that H acts fiberwise, the quotient X/H is a fiber bundle over B .

Proof. It will suffice to give local trivializations respecting the H action. Start with any local trivialization of the fiber bundle $\mathbb{P}^n \times U \rightarrow U$, $b \in U$. We can consider the trivial families of algebraic groups $H \times U$, $PGL_n \times U$, and $SL_n \times U$. The action of H on $X|_U$ determines a map $H \times U \rightarrow PGL_n \times U$, and we can take the fiber product

$$\begin{array}{ccc} \mathcal{H} & \longrightarrow & SL_n \times U \\ \downarrow & & \downarrow \\ H \times U & \longrightarrow & PGL_n \times U \end{array}$$

Each fiber of \mathcal{H} over U will be an extension of H by \mathbb{Z}/n , and since the moduli of finite groups is discrete, they will all be isomorphic to a fixed group \tilde{H} . After possibly an etale base change to V , we can trivialize \mathcal{H} to obtain

$$\begin{array}{ccc} \tilde{H} \times V & \longrightarrow & SL_n \times V \\ \downarrow & & \downarrow \\ H \times V & \longrightarrow & PGL_n \times V \end{array}$$

So we have an action of \tilde{H} on the bundle $Y = \mathbb{C}^n \times V \rightarrow V$, the representation R on each fiber isomorphic because taking the character on each fiber is a continuous function on V . Let Z be the bundle $\mathbb{C}^n \times V \rightarrow V$ with \tilde{H} acting on Z via the representation R on the first factor. By Schur's lemma the vector bundle $\text{Hom}(Z, Y)$ is a line bundle on V . Take a nonzero section, and let W be the open set on which it does not vanish. Over W , Z gives a trivialization of Y for which the \tilde{H} action is via only the first factor. Projectivizing, we obtain a trivialization of X over W on which H acts via only the first factor, as desired. \square

2.2.3 Computation of the virtual Poincaré polynomial of U_d

Lemma 2.10. *$R^j \pi_* \mathbb{Q}_{\tilde{U}_d}$ is a trivial local system on $\tilde{M}_{1,1}$.*

Proof. We will show that $R^j \pi_* \mathbb{Q}_{\tilde{U}_d}$ is trivial by demonstrating a nonvanishing global section. A nonvanishing global section of $R^j \pi_* \mathbb{Q}_{\tilde{U}_d} = R^j \pi_* \mathbb{Q}_{\tilde{P}_d/\hat{H}/\hat{G}}$ is simply a nonvanishing \hat{H} -invariant \hat{G} -equivariant global section of $R^j \pi_* \mathbb{Q}_{\tilde{P}_d}$.

\tilde{P}_d is the projectivization of a vector bundle over $\tilde{M}_{1,1}[d]$, hence is Zariski locally trivial. So we can cover \tilde{P}_d by open sets Ω over which there is a trivialization $\tilde{P}_d|_{\Omega} = \Omega \times \mathbb{P}^{d-1}$. For any such Ω , fix a linear subspace $\Lambda \subset \mathbb{P}^{d-1}$ of dimension j . Under the trivialization, this gives a subvariety $\lambda := \Lambda \times \Omega$ of $\tilde{P}_d|_{\Omega}$. The union $\lambda^{\hat{H}} := \sum_{h \in \hat{H}} h \cdot \lambda$ is \hat{H} -invariant, and its class $[\lambda^{\hat{H}}] \in H^j(P_d|_{\Omega})$ is \hat{H} -invariant and \hat{G} -equivariant. Because the action of \hat{H} is holomorphic on \mathbb{P}^{d-1} , it is in particular orientation preserving and hence for each point $t \in \Omega$, the class $[\lambda^{\hat{H}}(t)]$ is a nonzero element of $H^j(\mathbb{P}^{d-1}, \mathbb{Q})^{\hat{H}}$.

In this way we obtain a section $s_{\Omega} \in R^j \pi_* \mathbb{Q}_{\tilde{P}_d}(\Omega)$ for each Ω in the cover. For Ω, Ω' two open sets of the cover, the sections s_{Ω} and $s_{\Omega'}$ agree on the intersection, so this defines a nonvanishing \hat{H} -invariant \hat{G} -equivariant global section of $R^j \pi_* \mathbb{Q}_{\tilde{P}_d}$, as desired. \square

Proposition 2.11. $P_{U_d}^{vir}(t) = (1 + \dots + t^{2d-2}) \cdot P_{M_{1,1}}^{vir}(t) = (t^2 + \dots + t^{2d})$.

Proof. By the additivity of the virtual Poincaré polynomial on locally closed stratifications,

$$P_{U_d}^{vir}(t) = P_{\widetilde{U}_d}^{vir}(t) + P_{U_d|_{M_{1,1} \setminus \widetilde{M}_{1,1}}}^{vir}(t).$$

Because we've shown that the local system $R^j \pi_* \mathbb{Q}_{\widetilde{U}_d}$ is trivial, we can apply Lemma 2.1 to see that $P_{\widetilde{U}_d}^{vir}(t) = P_{\mathbb{P}^{d-1}/\hat{H}}^{vir}(t) \cdot P_{\widetilde{M}_{1,1}}^{vir}(t)$, and by Lemma 2.2, this is $(1 + \dots + t^{2d-2})P_{\widetilde{M}_{1,1}}^{vir}(t)$.

Meanwhile, the complement $M_{1,1} \setminus \widetilde{M}_{1,1}$ consists of two points ξ_4 and ξ_6 , and $U_d|_{\xi_4}$ (*resp.* $U_d|_{\xi_6}$) is \mathbb{P}^{d-1}/\hat{H} further quotiented by $\mathbb{Z}/2$ (*resp.* $\mathbb{Z}/3$). In any case, by Lemma 2.2, the virtual Poincaré polynomial of these fibers is also $(1 + \dots + t^{2d-2})$, and we conclude that

$$P_{U_d}^{vir} = (1 + \dots + t^{2d-2})P_{\widetilde{M}_{1,1}}^{vir}(t) + (1 + \dots + t^{2d-2})P_{M_{1,1} \setminus \widetilde{M}_{1,1}}^{vir}(t) = (1 + \dots + t^{2d-2})P_{M_{1,1}}^{vir}(t)$$

as desired. \square

Note that Proposition 2.11 holds for all d , as the expressions for $d = 1$ and $d = 2$ given in Equations (2) and (3) match this formula.

2.3 Boundary strata

Now we compute the virtual Poincaré polynomial of the strata Δ_d^i . Recall that Δ_d^i is the moduli space parameterizing sets D of d unordered possibly coincident points on C_i^{ms} , the nonsingular locus of C_i . D is isomorphic to D' if there is an automorphism $\phi : C_i \rightarrow C_i$ such that $\phi(D) = D'$. Because we work with objects modulo isomorphisms, in the remainder let us consider C_i to be rigidified by the requirement that the two nodes on each component be $\{0, \infty\} \in \mathbb{P}^1$, and one component meets the next by gluing 0 to ∞ . With this data fixed, the automorphism group of C_i is $(\mathbb{C}^*)^i \rtimes D_i$, where $(\mathbb{C}^*)^i$ acts by scaling on each component and D_i acts by the standard representation on an i -gon in the plane.

We further stratify Δ_d^i into substrata defined by fixing, modulo the action of D_i , the number of points on each component of C_i . Let $\Delta_d^i[d_1, \dots, d_i]$ denote the subset of Δ_d^i with d_1 points on one component, d_2 points on the next, and so on. We will show that $P_{\Delta_d^i[d_1, \dots, d_i]}^{vir}(t) = t^{2(d-i)}$ and count the number of such substrata.

Lemma 2.12. $P_{\Delta_d^i[d_1, \dots, d_i]}^{vir}(t) = t^{2(d-i)}.$

Proof. The simplest case is Δ_d^1 . Here C_1 is a nodal \mathbb{P}^1 and there is only one substratum

$$\Delta_d^1[d] \simeq \text{Sym}^d(\mathbb{P}^1 \setminus \{0, \infty\})/(\mathbb{C}^* \rtimes \mathbb{Z}/2).$$

Under the isomorphism $\text{Sym}^d(\mathbb{C}^*) \simeq \mathbb{C}^{d-1} \times \mathbb{C}^*$ given by $\{a_1, \dots, a_d\} \rightarrow (b_0, \dots, b_{d-1})$, where $z^d + b_{d-1}z^{d-1} + \dots + b_0 = (z - a_1)\dots(z - a_d)$, the action of $\mathbb{C}^* \rtimes \mathbb{Z}/2$ is

$$(\lambda, \tau)(b_0, \dots, b_{d-1}) = \begin{cases} (\lambda^d b_0, \dots, \lambda b_{d-1}) & \text{if } \tau = id \in \mathbb{Z}/2 \\ (\frac{1}{\lambda^d b_0}, \frac{b_{d-1}}{\lambda^{d-1} b_0}, \dots, \frac{b_1}{\lambda b_0}) & \text{if } \tau \neq id \in \mathbb{Z}/2 \end{cases}$$

So

$$\Delta_d^i[d] \simeq \text{Sym}^d(\mathbb{C}^*)/(\mathbb{C}^* \rtimes \mathbb{Z}/2) = (\text{Sym}^d(\mathbb{C}^*)/\mathbb{C}^*)/(\mathbb{Z}/2) = (\mathbb{C}^{d-1}/\mathbb{Z}/d)/(\mathbb{Z}/2)$$

and we conclude that $P_{\Delta_d^1[d]}^{vir}(t) = t^{2(d-1)}$ by Lemma 2.3.

In the general case,

$$\begin{aligned} \Delta_d^i[d_1, \dots, d_i] &\simeq \left(\text{Sym}^{d_1}(\mathbb{C}^*) \times \dots \times \text{Sym}^{d_i}(\mathbb{C}^*) \right) / (\mathbb{C}^*)^i \rtimes D_i \\ &= \left(\text{Sym}^{d_1}(\mathbb{C}^*)/\mathbb{C}^* \times \dots \times \text{Sym}^{d_i}(\mathbb{C}^*)/\mathbb{C}^* \right) / D_i \\ &= \left(\text{Sym}^{d_1-1}(\mathbb{C}^*)/(\mathbb{Z}/d_1) \times \dots \times \text{Sym}^{d_i-1}(\mathbb{C}^*)/(\mathbb{Z}/d_i) \right) / D_i. \end{aligned}$$

Again, by Lemma 2.3 we conclude $P_{\Delta_d^i[d_1, \dots, d_i]}^{vir}(t) = t^{2(d-i)}$. □

It remains now only to compute the number of substrata of Δ_d^i . We do so by observing the following bijection between the substrata $\Delta_d^i[d_1, \dots, d_i]$ of Δ_d^i and necklaces of i black beads and $d-i$ white beads. The substrata of Δ_d^i are indexed by i -gons decorated with a choice of a number $d_j \geq 1$ for each side, d_j denoting the number of points of D on the corresponding component of C_i . Construct a necklace from such a decorated i -gon by going putting one black bead at every vertex and $d_j - 1$ white beads on each edge.

Let us denote by N_{d-i}^i the number of necklaces of i black beads and $d-i$ white beads. We conclude

Proposition 2.13. $P_{\coprod \Delta_d^i}^{vir}(t) = \sum_{i=1}^d N_{d-i}^i t^{2(d-i)}$.

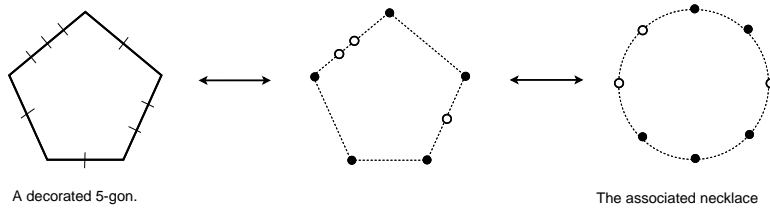


Figure 3: An example of the bijection between substrata $\Delta_d^i[d_1, \dots, d_i]$ and necklaces.

Combining this with the results of Section 2.2,

$$\begin{aligned}
P_{\overline{Q}_1(G(1,1),d)}^{vir}(t) &= (t^2 + \dots + t^{2d}) + \sum_{i=1}^d N_{d-i}^i t^{2(d-i)} \\
&= (t^2 \dots + t^{2d-2}) + \sum_{i=0}^d N_{d-i}^i t^{2(d-i)} \\
&= 1 + 2t^2 + \dots + 2t^{2d-2} + t^{2d}.
\end{aligned}$$

Note that the necklace counting problem is symmetric under the interchange of the black and white beads. That is, $N_{d-i}^i = N_i^{d-i}$. By Poincaré Duality, we know the Poincaré polynomial of $\overline{Q}_1(G(1,1),d)$ will satisfy

$$P_{\overline{Q}_1(G(1,1),d)}^{vir}(t) = \left(P_{\overline{Q}_1(G(1,1),d)}^{vir}(t) \right)^{-1} \cdot t^{2d}.$$

In this particular case, switching black and white beads is the explicit manifestation of this expected symmetry.

The question of counting necklaces of colored beads is a classical one in combinatorics. Such necklaces are named "Polya's necklaces" in honor of Polya who gave a formula for the generating function for N_{d-i}^i (also given independently by Redfield) [Po37], [?]. Using his formula we conclude

Theorem 2.14. *If d is odd,*

$$P_{\overline{Q}_1(G(1,1),d)}(t) = P_{\overline{Q}_1(G(1,1),d)}^{vir}(t) = \left\{ \frac{1}{2d} \left[d(1+t^2)(1+t^4)^{\frac{d-1}{2}} + \sum_{k|d} \phi\left(\frac{d}{k}\right)(1+t^{\frac{2d}{k}})^k \right] + \sum_{j=1}^{d-1} t^{2j} \right\}$$

and if d is even,

$$P_{\overline{Q}_1(G(1,1),d)}(t) = P_{\overline{Q}_1(G(1,1),d)}^{vir}(t) = \frac{1}{2d} \left[\frac{d}{2}(1+t^2)^2(1+t^4)^{\frac{d}{2}-2} + \frac{d}{2}(1+t^4)^{\frac{d}{2}} + \sum_{k|d} \phi\left(\frac{d}{k}\right)(1+t^{\frac{2d}{k}})^k \right] + \sum_{j=1}^{d-1} t^{2j}$$

3 Topology of $\overline{Q}_1(\mathbb{P}^{n-1}, d)$

3.1 Introduction to the Białynicki-Birula stratification

Let X be a smooth projective variety with a \mathbb{C}^* action. The fixed locus $X^{\mathbb{C}^*}$ is smooth and hence the disjoint union of irreducible components $X_1 \cup \dots \cup X_r$ [I72]. Białynicki-Birula [B74] gave a

natural stratification of X indexed by fixed loci X_i , namely the cell X_i^+ associated to X_i is defined as

$$X_i^+ = \{x \in X \mid \lim_{\lambda \rightarrow 0} \lambda x \in X_i\}.$$

An alternate description is by considering the normal bundle N_i to X_i . N_i decomposes into a direct sum

$$N_i = \bigoplus_{m \in \mathbb{Z}} N_i(m)$$

where $N_i(m)$ is the subbundle of N_i composed of semi-invariants of weight m . Then X_i^+ is isomorphic to the total space of the subbundle $\bigoplus_{m > 0} N_i(m)$. Let $d_i(m) = \dim N_i(m)$, $d_i^+ = \sum_{m > 0} d_i(m)$.

Theorem 3.1. (*[B74] Theorem 1*) *If X is a smooth projective variety with a \mathbb{C}^* action, then*

$$P_X(t) = \sum_{i=1}^r P_{X_i}(t) t^{2d_i^+}.$$

It is natural to extend this result to singular varieties [CG83], and in fact Theorem 3.1 can be extended to the case that X is a smooth Deligne-Mumford stack, giving a tool for computing the Betti numbers of such stacks. Specifically,

Theorem 3.2. (*[F06] Proposition 1.*) *Let X be a smooth projective orbifold with a \mathbb{C}^* -action, let F be the fixed locus and let F_i denote its connected components. Then*

(i) *X is the disjoint union of locally closed subvarieties S_i such that every S_i retracts onto the corresponding F_i and*

$$\overline{S_i} \subseteq \bigcup_{j \geq i} S_j$$

(ii) *the Betti numbers of X are*

$$h^m(X) = \sum_i h^{m-2n_i}(F_i)$$

where n_i is the codimension of S_i .

Note that for all Deligne-Mumford stacks with a coarse moduli scheme, the cohomology groups with \mathbb{Q} coefficients of the stack and the scheme are equal.

3.2 \mathbb{C}^* fixed loci of $\overline{Q}_1(\mathbb{P}^{n-1}, d)$

To use Theorem 3.2 to compute the Poincaré polynomial of $\overline{Q}_1(\mathbb{P}^{n-1}, d)$, we begin by fixing an action of \mathbb{C}^* on $\overline{Q}_1(\mathbb{P}^{n-1}, d)$ and describing the fixed loci of that action. Fix a general $(w_0, \dots, w_{n-1}) \in \mathbb{Z}^n$ (in particular such that w_i have no common factors) satisfying $w_0 < w_1 < \dots < w_{n-1}$. Let \mathbb{C}^* act on \mathbb{C}^n via

$$\lambda \cdot (x_0, \dots, x_{n-1}) = (\lambda^{w_0} x_0, \dots, \lambda^{w_{n-1}} x_{n-1}).$$

This induces an action on $\overline{Q}_1(\mathbb{P}^{n-1}, d)$ by taking this action on $\mathbb{C}^n \otimes \mathcal{O}_C$ of each stable quotient.

The fixed loci for this action on any stable quotient space $\overline{Q}_g(Gr(k, n), d)$ are described in [MOP09]. We record their result in the genus 1 rank 1 case here, with simplified notation. The fixed loci of $\overline{Q}_1(G(1, n), d)$ come in two types.

Type A.

There are n fixed loci isomorphic to $\overline{Q}_1(G(1, 1), d)$. They are $X_\ell := i_\ell(\overline{Q}_1(G(1, 1), d))$ for $0 \leq \ell \leq n-1$, where the inclusion $i_\ell : \overline{Q}_1(G(1, 1), d) \rightarrow \overline{Q}_1(\mathbb{P}^{n-1}, d)$ is defined by

$$i_\ell : (C, 0 \rightarrow S \rightarrow \mathcal{O} \rightarrow Q \rightarrow 0) \rightarrow \left(C, 0 \rightarrow S \rightarrow \begin{array}{c} \mathcal{O}^\ell \\ \oplus \\ \mathcal{O} \\ \oplus \\ \mathcal{O}^{n-\ell-1} \end{array} \rightarrow \begin{array}{c} \mathcal{O}^\ell \\ \oplus \\ Q \\ \oplus \\ \mathcal{O}^{n-\ell-1} \end{array} \rightarrow 0 \right). \quad (4)$$

The normal bundle N_ℓ to X_ℓ is $Hom(S, \mathcal{O}^\ell \oplus \mathcal{O}^{n-\ell-1})$ and the subbundle of positive weight is $N_\ell^+ = Hom(S, \mathcal{O}^\ell)$, with dimension

$$d_\ell^+ = d\ell. \quad (5)$$

Type B.

These fixed loci are indexed by the following decorated graphs:

- Γ : an m -cycle. Let the vertices be labelled $\{v_i | 1 \leq i \leq m\}$ and the edges $\{e_i | 1 \leq i \leq m\}$ where e_i is the edge from v_i to v_{i+1} .
- $\nu : \{\text{vertices}\} \rightarrow \{\mathbb{C}^* \text{ fixed points in } \mathbb{P}^{n-1}\}$
- $s : \{\text{vertices}\} \rightarrow \mathbb{N} \cup \{0\}$
- $\delta : \{\text{edges}\} \rightarrow \{\text{positive integers}\}$

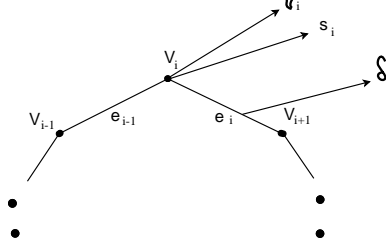


Figure 4: Γ and its decorations

subject to the constraints

$$\nu(v_i) \neq \nu(v_{i+1}) \quad (6)$$

with the indices taken modulo m , i.e. $\nu(v_m) \neq \nu(v_1)$, and

$$\sum_i s(v_i) + \delta(e_i) = d. \quad (7)$$

To fix notation, name the \mathbb{C}^* fixed points in \mathbb{P}^{n-1} p_0, \dots, p_{n-1} where $p_j = [0 : \dots : 1 : \dots : 0]$ with all but the j^{th} coordinate equal to 0. If $\nu(v_i) = p_j$, we will write $\nu_i = j$. Similarly, let s_i denote $s(v_i)$ and δ_i denote $\delta(e_i)$. The value of s on a vertex will record the amount of torsion on the curve associated to that vertex and the value of δ on an edge will be the covering number for the map from the associated \mathbb{P}^1 .

Fix such a decorated graph Γ . Let μ be the number of vertices for which $s_i > 0$. The points of the associated fixed locus X_Γ are certain stable quotients on $C \simeq C_{m+\mu}$, the cycle of $m + \mu$ rational curves. C consists of components $\{C_{v_i}\}$ for i such that $s_i > 0$ and $\{C_{e_i}\}$ for all i , glued by the graph incidences. Let $C' \simeq C_m$ be the m -cycle of rational curves obtained by gluing only the components $\{C_{e_i}\}$ along the graph incidences.

X_Γ is isomorphic to $\prod_{i|s_i>0} \overline{M}_{0,2|s_i}/Aut(\Gamma)$. The isomorphism $\Psi : \prod_{i|s_i>0} \overline{M}_{0,2|s_i}/Aut(\Gamma) \rightarrow X_\Gamma$ is given as follows. Fix a point $\zeta \in \prod_{i|s_i>0} \overline{M}_{0,2|s_i}/Aut(\Gamma)$. For each i such that $s_i > 0$, ζ specifies marked points $\{p_1, p_2\} \cup \{q_1 \dots q_{s_i}\}$ on $C_{v_i} \simeq \mathbb{P}^1$. The stable quotient on the component C_{v_i} is

$$0 \rightarrow \mathcal{O}_{C_{v_i}}(-q_1 \dots - q_{s_i}) \rightarrow \mathbb{C}^n \otimes \mathcal{O}_{C_{v_i}} \rightarrow Q \rightarrow 0$$

where the inclusion is into the factor of $\mathbb{C}^n \otimes \mathcal{O}_{C_{v_i}}$ determined by the \mathbb{C}^* -fixed point of \mathbb{P}^{n-1} associated to v_i . Namely, if $\nu(v_i) = p_j$ then $\mathcal{O}_{C_{v_i}}(-q_1 \dots - q_{s_i})$ injects into the j^{th} copy of $\mathcal{O}_{C_{v_i}}$.

The stable quotient on the component C_{e_i} is obtained as follows. Let $L \simeq \mathbb{P}^1$ be the line in \mathbb{P}^{n-1}

between $\nu(v_i)$ and $\nu(v_{i+1})$. Let

$$f_i : C_{e_i} \simeq \mathbb{P}^1 \rightarrow L$$

be the map of degree $\delta(e_i)$ ramified over completely over $\nu(v_i)$ and $\nu(v_{i+1})$. The stable quotient on C_{e_i} is obtained by pulling back the tautological sequence of \mathbb{P}^{n-1} along f_i .

By construction the stable quotients on the components C_{v_i} and C_{e_i} are compatible so they can be glued to obtain a stable quotient on C , and this is the \mathbb{C}^* -fixed point $\Psi(\zeta)$.

3.3 Białyński-Birula Cells

To use the above \mathbb{C}^* action to compute the Poincaré polynomial of $\overline{Q}_1(\mathbb{P}^{n-1}, d)$, we must compute the fiber dimension of each cell. To compute the fiber dimension d_Γ^+ of the Białyński-Birula affine bundle over X_Γ , it suffices to take any point $x \in X_\Gamma$ and compute the weights of the \mathbb{C}^* action on the normal bundle N_Γ to X_Γ at x . d_Γ^+ is the number of positive weights of this representation.

First consider the case that all $s_i = 0$ in the decorated graph Γ . Then the associated fixed locus is a point and the associated stable quotient is the pullback of the tautological sequence of \mathbb{P}^{n-1} along the map $f : C \rightarrow \mathbb{P}^{n-1}$ described above, where $C \simeq C_m$ with components $\{C_{e_i}\}$. In this case the deformation theory is just that for stable maps, as described in [Ho03]. The normal bundle in this case is just the tangent space to the point, which in K-theory is

$$-Aut(C) + \bigoplus_i H^0(f_i^* T_{\mathbb{P}^{n-1}}) - \bigoplus_i H^0((f^* T_{\mathbb{P}^{n-1}})|_{v_i}) + Def(C).$$

Note in this case the curves $C = C'$ are equal.

In the general case, Marian-Oprea-Pandharipande compute that for each nonzero s_i the tangent space has an additional summand of

$$\bigoplus_{j \neq \nu(i)} H^0(\mathcal{O}_C(\sigma_i)|_{\sigma_i}) \otimes [w_j - w_{\nu(i)}]$$

where $\sigma_i \subset C_{v_i}$ is the divisor of torsion points corresponding to s_i and j runs from 0 to $n-1$. $H^0(\mathcal{O}_C(\sigma_i)|_{\sigma_i})$ carries the trivial \mathbb{C}^* action and $[w_j - w_{\nu(i)}]$ denotes the vector space \mathbb{C} with the representation $\lambda \cdot v = \lambda^{w_j - w_{\nu(i)}} v$ [MOP09].

We conclude that in K-theory, the fiber of N_Γ at x is

$$-Aut(C') + \bigoplus_i H^0(f_i^* T_{\mathbb{P}^{n-1}}) - \bigoplus_i H^0((f^* T_{\mathbb{P}^{n-1}})|_{v_i}) + Def(C) + \bigoplus_i \bigoplus_{j \neq \nu(i)} H^0(\mathcal{O}_C(\sigma_i)|_{\sigma_i}) \otimes [w_j - w_{\nu(i)}]. \quad (8)$$

Arbitrary subsums of this expression in K-theory will not necessarily correspond to vector spaces, but the following decomposition does correspond to a decomposition into three vector spaces: $N_\Gamma|_x = N_1 \oplus N_2 \oplus N_3$, where

$$\begin{aligned}
N_1 &= \bigoplus_i N_1(i) = \bigoplus_i [H^0(f_i^* T_{\mathbb{P}^{n-1}}) - H^0((f^* T_{\mathbb{P}^{n-1}})|_{v_{i+1}}) - \text{Aut}(C_{e_i})] \\
N_2 &= \bigoplus_i N_2(i) = \text{Def}(C) \\
N_3 &= \bigoplus_i N_3(i) = \bigoplus_i \bigoplus_{j \neq i} H^0(\mathcal{O}_C(\sigma_i)|_{\sigma_i}) \otimes [w_j - w_i].
\end{aligned} \tag{9}$$

The \mathbb{C}^* weights of $N_1(i)$ are

$$\left\{ \begin{array}{ll} \frac{c}{\delta_i}(w_{\nu_i} - w_{\nu_{i+1}}) & \text{for all } \begin{array}{l} -\delta_i \leq c \leq -1 \\ 1 \leq c \leq \delta_i - 1 \end{array} \\ \text{and} \\ w_j - (\frac{\delta_i - c}{\delta_i} w_{\nu_i} + \frac{c}{\delta_i} w_{\nu_{i+1}}) & \text{for all } \begin{array}{l} j \neq \nu_i \text{ or } \nu_{i-1} \\ 0 \leq c \leq \delta_i - 1. \end{array} \end{array} \right\} \tag{10}$$

The vector space $N_2(i)$ is the deformation of C given by smoothing the node v_i if $s_i = 0$ or smoothing the two nodes on C_{v_i} if $s_i \neq 0$. The \mathbb{C}^* weights of $N_2(i)$ are thus

$$\left\{ \begin{array}{ll} w_{\nu_{i+1}} + w_{\nu_{i-1}} - 2w_{\nu_i} & \text{if } s_i = 0 \\ \text{or} \\ w_{\nu_{i+1}} - w_{\nu_i} \text{ and } w_{\nu_{i-1}} - w_{\nu_i} & \text{if } s_i \neq 0. \end{array} \right\} \tag{11}$$

Finally, the \mathbb{C}^* weights of $N_3(i)$ are easiest to calculate, and are

$$\{ w_j - w_i \text{ with multiplicity } s_i, \quad \text{for all } j \neq i. \} \tag{12}$$

So the set of all weights of the normal bundle to X_Γ are given by the expressions (10), (11), (12) as i runs from 1 to m . The dimension of the fibers of the affine bundle X_Γ^+ over X_Γ is d_Γ^+ , the number of positive weights.

3.4 Computation of Betti numbers of $\overline{\mathcal{Q}}_1(\mathbb{P}^{n-1}, d)$

Marian-Oprea-Pandharipande proved that $\overline{\mathcal{Q}}_1(\mathbb{P}^{n-1}, d)$ is a smooth Deligne-Mumford stack whose coarse moduli space is a scheme [MOP09], thus we can apply Theorem 3.2 to compute the Betti numbers of $\overline{\mathcal{Q}}_1(\mathbb{P}^{n-1}, d)$ by the formula

$$P_X(t) = \sum_{i=1}^r P_{X_i}(t) t^{2d_i^+}.$$

For small n, d , we wrote down all the fixed loci and with the aid of some Mathematica script computed d_Γ^+ for all of them. (For $n = d = 2$, there was 1 fixed locus of type B and 4 weights to compute for that locus. For $n = d = 4$, there were ~ 100 fixed loci of type B and ~ 1600 weights.) The result is:

$$\begin{aligned}
P_{\overline{Q}_1(\mathbb{P}^1, 2)}(t) &= 1 + 2t^2 + 3t^4 + 2t^6 + t^8 \\
P_{\overline{Q}_1(\mathbb{P}^2, 3)}(t) &= 1 + 2t^2 + 3t^4 + 4t^6 + 4t^8 + 4t^{10} + 4t^{12} + 3t^{14} + 2t^{16} + t^{18} \\
P_{\overline{Q}_1(\mathbb{P}^3, 4)}(t) &= \\
1 + 2t^2 + 4t^4 + 5t^6 + 9t^8 + 10t^{10} + 14t^{12} + 14t^{14} + 17t^{16} + 14t^{18} + 14t^{20} + 10t^{22} + 9t^{24} + 5t^{26} + 4t^{28} + 2t^{30} + t^{32}.
\end{aligned} \tag{13}$$

While it is difficult to compute the full Poincaré polynomial of $\overline{Q}_1(\mathbb{P}^{n-1}, d)$ for arbitrary n, d because of the large number of fixed loci, it is nonetheless worthwhile to compute the most accessible Betti numbers, namely $h^i(\overline{Q}_1(\mathbb{P}^{n-1}, d), \mathbb{Q})$ for $i = 2$ and i odd.

In the case of $h^2(\overline{Q}_1(\mathbb{P}^{n-1}, d), \mathbb{Q})$, the main observation is

Lemma 3.3. *For any fixed locus X_Γ in $\overline{Q}_1(\mathbb{P}^{n-1}, d)$ of type B, $d_\Gamma^+ \geq 2$.*

Proof. Fix a decorated graph Γ . Pick a vertex v_i of Γ such that ν_i is minimal. We demonstrate two positive weights of the bundle N_Γ . The first is the weight $w_{\nu_{i+1}} - w_{\nu_i}$ of $N_1(i)$, which appears in the first expression of (10) for $c = \delta_i$. This is positive because by our original choice of \mathbb{C}^* action, if ν_i is minimal then $w_{\nu_i} < w_{\nu_{i+1}}$.

The second is a weight of $N_2(i)$. Here there are two cases. If $s_i = 0$, we have the weight $w_{\nu_{i+1}} + w_{\nu_{i-1}} - 2w_{\nu_i}$. If $s_i > 0$ we have two weights $w_{\nu_{i-1}} - w_{\nu_i}$ and $w_{\nu_{i+1}} - w_{\nu_i}$. This gives either one or two positive weights, again by minimality of w_{ν_i} . In either case there are at least two positive weights of the normal bundle N_Γ , and we conclude $d_\Gamma^+ \geq 2$. \square

Proposition 3.4. $h^2(\overline{Q}_1(\mathbb{P}^{n-1}, d), \mathbb{Q}) = 2$ for all n, d .

Proof. We consider the cases $d = 1$ and $d \geq 2$ separately. When $d = 1$,

$$\overline{Q}_1(\mathbb{P}^{n-1}, 1) \simeq \overline{M}_{1,1} \times \mathbb{P}^{n-1} \simeq \mathbb{P}^1 \times \mathbb{P}^{n-1} \text{ ([MOP09] Section 2)}.$$

We see directly in this case that $h^2(\mathbb{P}^1 \times \mathbb{P}^{n-1}) = 2$.

When $d \geq 2$: For fixed loci of type A, which we'd named X_ℓ for $0 \leq \ell \leq n-1$, we've seen $d_\ell^+ > 2$ for all $\ell \geq 1$ (5). Together with Lemma 3.3 this implies that in the expression

$$P_{\overline{Q}_1(\mathbb{P}^{n-1}, d)}(t) = \sum_i P_{X_i}(t) t^{2d_i^+}$$

given by Theorem 3.2 there is only one fixed locus that contributes to the t^2 term, namely $X_0 \simeq \overline{Q}_1(G(1, 1), d)$, because the Bialinicki-Birula cell associated to X_0 has zero dimensional fibers. By Theorem 2.14, $P_{\overline{Q}_1(G(1, 1), d)} = 1 + 2t^2 + \dots + 2t^{2d-2} + t^{2d}$. Therefore

$$h_2(\overline{Q}_1(\mathbb{P}^{n-1}, d), \mathbb{Q}) = h_2(\overline{Q}_1(G(1, 1), d)) = 2.$$

By Poincaré duality, $h^2(\overline{Q}_1(\mathbb{P}^{n-1}, d), \mathbb{Q})$ also equals 2. \square

To compute $h^i(\overline{Q}_1(\mathbb{P}^{n-1}, d), \mathbb{Q})$ when i is odd, we will use a refinement of the following Lemma from [MOP09].

Lemma 3.5. ([MOP09], Lemma 2). *For $d > 0$, the Poincaré polynomial of $\overline{M}_{0,2|d}/S_d$ is*

$$P_{\overline{M}_{0,2|d}/S_d}(t) = (1 + t^2)^{d-1}.$$

Proposition 3.6. *For all n, d , and all i odd, $h^i(\overline{Q}_1(\mathbb{P}^{n-1}, d), \mathbb{Q}) = 0$. That is, the odd rational cohomology of $\overline{Q}_1(\mathbb{P}^{n-1}, d)$ vanishes.*

Proof. By Theorem 3.2 $P_{\overline{Q}_1(\mathbb{P}^{n-1}, d)}(t) = \sum P_{X_i}(t)t^{2d_i^+}$, so if the Poincaré polynomial of every fixed locus vanishes in odd degree, $P_{\overline{Q}_1(\mathbb{P}^{n-1}, d)}(t)$ will too. Recall that there are two types of fixed loci. A fixed locus X_ℓ of the first type is isomorphic to $\overline{Q}_1(G(1, 1), d)$, and by Theorem 2.14 the Poincaré polynomial of X_ℓ vanishes in odd degree.

A fixed locus of the second type X_Γ is isomorphic to some

$$\prod_{i|s_i > 0} (\overline{M}_{0,2|s_i}/S_{s_i})/Aut(\Gamma),$$

where we take the convention that the empty product is a point. Lemma 3.5, does not compute the Poincaré polynomial directly if $Aut(\Gamma)$ is not trivial. But in that case, a slightly modified version of the proof of Lemma 3.5 of [MOP09] adding the use of Lemma 2.3 gives

$$P_{X_\Gamma}^{vir}(t) = \prod_{i|s_i > 0} (1 + t^2)^{s_i-1}.$$

We conclude that for all fixed loci the Poincaré polynomial vanishes in odd degree, so $P_{\overline{Q}_1(\mathbb{P}^{n-1}, d)}(t)$ also vanishes in odd degree. \square

In [F06], Fontanari proves a similar result for the Vakil-Zinger desingularization $\widetilde{M}_{1,n}^0(\mathbb{P}^r, d)$ of $\overline{M}_{1,n}(\mathbb{P}^r, d)$:

Theorem 3.7. ([F06], Theorem 1.) *$H^k(\widetilde{M}_{1,n}^0(\mathbb{P}^r, d), \mathbb{Q}) = 0$ for every $n \geq 0$, $r \geq 1$, and odd $k < 10$.*

4 Picard group

We use the exponential sequence to parlay our knowledge about the singular cohomology of $\overline{Q}_1(\mathbb{P}^{n-1}, d)$ into knowledge of $\text{Pic}(\overline{Q}_1(\mathbb{P}^{n-1}, d))_{\mathbb{Q}}$. Taking homology of the exponential sequence for $\overline{Q}_1(\mathbb{P}^{n-1}, d)$ and tensoring by \mathbb{Q} gives:

$$H^1(\overline{Q}_1(\mathbb{P}^{n-1}, d), \mathcal{O}) \otimes \mathbb{Q} \rightarrow \text{Pic}(\overline{Q}_1(\mathbb{P}^{n-1}, d)) \otimes \mathbb{Q} \rightarrow H^2(\overline{Q}_1(\mathbb{P}^{n-1}, d), \mathbb{Z}) \otimes \mathbb{Q}.$$

By Proposition 3.6, $h^1(\overline{Q}_1(\mathbb{P}^{n-1}, d), \mathbb{Q}) = 0$ so

$$H^1(\overline{Q}_1(\mathbb{P}^{n-1}, d), \mathcal{O}) = H^{0,1}(\overline{Q}_1(\mathbb{P}^{n-1}, d)) \subset H^1(\overline{Q}_1(\mathbb{P}^{n-1}, d), \mathbb{C}) = 0.$$

Hence $\text{Pic}(\overline{Q}_1(\mathbb{P}^{n-1}, d)) \otimes \mathbb{C}$ injects into $H^2(\overline{Q}_1(\mathbb{P}^{n-1}, d), \mathbb{C})$, and its image is $H^{1,1}(\overline{Q}_1(\mathbb{P}^{n-1}, d))$. Since $\text{Pic}(\overline{Q}_1(\mathbb{P}^{n-1}, d)) \otimes \mathbb{C}$ is nontrivial and

$$H^{2,0}(\overline{Q}_1(\mathbb{P}^{n-1}, d)) \oplus H^{1,1}(\overline{Q}_1(\mathbb{P}^{n-1}, d)) \oplus H^{0,2}(\overline{Q}_1(\mathbb{P}^{n-1}, d)) = H^2(\overline{Q}_1(\mathbb{P}^{n-1}, d), \mathbb{C}) \simeq \mathbb{C}^2,$$

we conclude that $H^{2,0}(\overline{Q}_1(\mathbb{P}^{n-1}, d)) = H^{0,2}(\overline{Q}_1(\mathbb{P}^{n-1}, d)) = 0$ and

$$\text{Pic}(\overline{Q}_1(\mathbb{P}^{n-1}, d)) \otimes \mathbb{Q} \simeq H^2(\overline{Q}_1(\mathbb{P}^{n-1}, d), \mathbb{Q}).$$

In particular, we have:

Theorem 4.1. *The Picard rank of $\overline{Q}_1(\mathbb{P}^{n-1}, d)$ is equal to $h^2(\overline{Q}_1(\mathbb{P}^{n-1}, d), \mathbb{Q}) = 2$.*

4.1 Construction of Divisors

We begin by defining three divisors on $\overline{Q}_1(G(1, 1), d)$. We define the quotient map

$$\varphi : \overline{M}_{1,0|d} \rightarrow \overline{Q}_1(G(1, 1), d)$$

by

$$\varphi(C, p_1, \dots, p_d) = (C, p_1 + \dots + p_d).$$

Definition 4. *A point $(C, D) \in \overline{Q}_1(G(1, 1), d)$, where $D = p_1 + \dots + p_d$ is in D_b if any $p_i = p_j$, i.e. if D contains any point of multiplicity ≥ 2 .*

Definition 5. *Fix a complex number $\rho \in \mathbb{C}$. A point $(C, D) \in \overline{Q}_1(G(1, 1), d)$ is in $D_j(\rho)$ if the j -invariant of C is ρ .*

For any ρ and ρ' , $D_j(\rho)$ is linearly equivalent to $D_j(\rho')$. We will write simply D_j for the linear equivalence class.

For constructing the next divisor, we begin by defining a divisor $B_{\mathcal{P}_{a,b}}$ on $\overline{M}_{1,0|2}$.

Definition 6. Let \mathcal{P} be a pencil of plane cubics through 8 general points in \mathbb{P}^2 . Let a and b be two base points of this pencil. By the universal property of $\overline{M}_{1,0|2}$, this gives a map $\mathbb{P}^1 \rightarrow \overline{M}_{1,0|2}$. Let the curve $B_{\mathcal{P}_{a,b}}$ be the image of this map.

Since $\overline{M}_{1,0|2}$ is 2-dimensional, $B_{\mathcal{P}_{a,b}}$ is a divisor. To define the divisor D_{fd} , we will use the following maps:

$$\begin{array}{ccc} \overline{M}_{1,0|d} & \xrightarrow{\varphi} & \overline{Q}_1(G(1,1), d) \\ \pi \downarrow & & \\ \overline{M}_{1,0|2} & & \end{array}$$

where $\pi : \overline{M}_{1,0|d} \rightarrow \overline{M}_{1,0|2}$ is the projection given by $\pi : (C, q_1, \dots, q_d) \mapsto (\tilde{C}, q_1, q_2)$ where \tilde{C} is obtained from C by contracting any components that become unstable after forgetting the points q_3, \dots, q_d .

Definition 7.

$$D_{fd}(\mathcal{P}_{a,b}) := \varphi_*(\pi^*(B_{\mathcal{P}_{a,b}})).$$

$B_{\mathcal{P}_{a,b}}$ is a divisor in $\overline{M}_{1,0|2}$, so because φ is finite D_{fd} is a divisor in $\overline{Q}_1(G(1,1), d)$. Concretely, a point $(C, D) \in \overline{Q}_1(G(1,1), d)$ is in $D_{fd}(\mathcal{P}_{a,b})$ if there is an isomorphism $\rho : \tilde{C} \rightarrow \mathcal{P}_t$ from a component \tilde{C} of C to a fiber of the pencil such that a and b are both in the image $\rho(D|_{\tilde{C}})$. $D_{fd}(\mathcal{P}_{a,b})$ is so named because the pencil \mathcal{P} with the choice of a and b determines a set of fixed distances on each genus 1 curve.

For any two $\mathcal{P}_{a,b}$ and $\mathcal{P}'_{a',b'}$ $B_{\mathcal{P}_{a,b}}$ is linearly equivalent to $B_{\mathcal{P}'_{a',b'}}$ in $\overline{M}_{1,0|2}$ because one can explicitly construct a family over \mathbb{P}^1 of pencils of plane cubics with two -1 sections, such that $\mathcal{P}_{a,b}$ is one fiber of the family and $\mathcal{P}'_{a',b'}$ another. $D_{fd}(\mathcal{P}_{a,b})$ and $D_{fd}(\mathcal{P}'_{a',b'})$ are then also linearly equivalent and we will write simply D_{fd} for the linear equivalence class.

To define divisors on $\overline{Q}_1(\mathbb{P}^{n-1}, d)$ when $n > 1$, start with the open locus $Y \subset \overline{Q}_1(\mathbb{P}^{n-1}, d)$ of stable quotients which correspond to a map $f : C \rightarrow \mathbb{P}^{n-1}$ where C is a smooth genus 1 curve and the map is nondegenerate in the sense that the image $f(C)$ does not lie in any hyperplane. Now fix a hyperplane $H \subset \mathbb{P}^{n-1}$. There is a map

$$\Phi : Y \rightarrow \overline{Q}_1(G(1,1), d)$$

sending (C, f) to $(C, f(C) \cap H)$.

We define divisors on $\overline{\mathcal{Q}}_1(\mathbb{P}^{n-1}, d)$ by taking the preimage of divisors in $\overline{\mathcal{Q}}_1(G(1, 1), d)$ under Φ and then taking the closure. In our definition of the divisors below, we explicitly describe these closures. Note the similarity between this map from $H_2(\overline{\mathcal{Q}}_1(G(1, 1), d)) \rightarrow H_2(\overline{\mathcal{Q}}_1(\mathbb{P}^{n-1}, d))$ and the isomorphism described in [CG83] for a scheme X with a *good* \mathbb{C}^* action between the homology groups of X and the shifted sums of the homology groups of the fixed loci.

Where confusion is possible, we will put a superscript 1 to denote divisors in $\overline{\mathcal{Q}}_1(G(1, 1), d)$ and a superscript n to denote divisors in $\overline{\mathcal{Q}}_1(\mathbb{P}^{n-1}, d)$, but when it is clear what space we are working in, we will drop the superscripts. Also, for a section s of a line bundle, we will use $|s|$ to denote its divisor of zeros.

Definition 8. Fix a hyperplane $H \subset \mathbb{P}^{n-1}$, defined by $a_0x_0 + \dots + a_{n-1}x_{n-1} = 0$. Let $(C, [s_0, \dots, s_{n-1}])$ be a point in $\overline{\mathcal{Q}}_1(\mathbb{P}^{n-1}, d)$, and let $s_H = a_0s_0 + \dots + a_{n-1}s_{n-1}$. Let $D_b^n(H)$ be the subvariety of points $(C, [s_0, \dots, s_{n-1}])$ such that either $s_H = 0$ or on some component \tilde{C} the divisor $|s_H|$ contains a point with multiplicity ≥ 2 .

For any two hyperplanes H, H' in \mathbb{P}^{n-1} , $D_b^n(H)$ and $D_b^n(H')$ are linearly equivalent. We will write D_b^n for the linear equivalence class.

Definition 9. Fix a complex number $\rho \in \mathbb{C}$. A point $(C, [s_0, \dots, s_{n-1}]) \in \overline{\mathcal{Q}}_1(\mathbb{P}^{n-1}, d)$ is in $D_j^n(\rho)$ if the j -invariant of C is ρ .

As above, for any ρ and ρ' , $D_j^n(\rho)$ and $D_j^n(\rho')$ are linearly equivalent, and we will write D_j^n for the linear equivalence class.

Definition 10. Fix a hyperplane H . Let $\mathcal{P}_{a,b}$ be as in the definition of D_{fd}^1 . A point $(C, [s_0, \dots, s_{n-1}]) \in \overline{\mathcal{Q}}_1(\mathbb{P}^{n-1}, d)$ is in $D_{fd}^n(H, \mathcal{P}_{a,b})$ if either $s_H = 0$ or $(\tilde{C}, |s_H|) \in D_{fd}^1$. $((C, |s_H|))$ may not be stable, but after contracting unstable rational components to obtain \tilde{C} , $(\tilde{C}, |s_H|)$ will be stable, in $\overline{\mathcal{Q}}_1(G(1, 1), d)$.

As above, for any choices of $H, H', \mathcal{P}_{a,b}, \mathcal{P}'_{a',b'}$, $D_{fd}^n(H, \mathcal{P}_{a,b})$ is linearly equivalent to $D_{fd}^n(H', \mathcal{P}'_{a',b'})$. We will write simply D_{fd} for the linear equivalence class.

We will see that any two of these three divisors are not numerically equivalent, hence not linearly equivalent. So the line bundles associated to any pair of these divisors generates $\text{Pic}(\overline{\mathcal{Q}}_1(\mathbb{P}^{n-1}, d)) \otimes \mathbb{Q} = H^2(\overline{\mathcal{Q}}_1(\mathbb{P}^{n-1}, d), \mathbb{Q})$.

In Section 3.2, we introduced the inclusion $i_0 : \overline{\mathcal{Q}}_1(G(1, 1), d) \rightarrow \overline{\mathcal{Q}}_1(\mathbb{P}^{n-1}, d)$. The induced map $i_0^* : H^2(\overline{\mathcal{Q}}_1(\mathbb{P}^{n-1}, d), \mathbb{Q}) \rightarrow H^2(\overline{\mathcal{Q}}_1(G(1, 1), d), \mathbb{Q})$ sends D_j^n to D_j^1 and D_b^n to D_b^1 , hence is an isomorphism of vector spaces. Dually, the induced map $i_{0*} : H_2(\overline{\mathcal{Q}}_1(G(1, 1), d), \mathbb{Q}) \rightarrow H_2(\overline{\mathcal{Q}}_1(\mathbb{P}^{n-1}, d), \mathbb{Q})$ is an isomorphism.

4.2 Construction of Curves

We begin by defining curves in $\overline{\mathcal{Q}}_1(G(1, 1), d)$.

Definition 11. Fix a smooth genus 1 curve C and an effective divisor \tilde{D} on C of degree $d-1$. Let $(C, D) \in \overline{\mathcal{Q}}_1(G(1, 1), d)$ be in $\gamma_j(C, \tilde{D})$ if $D = \tilde{D} + p$ for some point $p \in C$.

In other words, $\gamma_j(C, \tilde{D})$ is obtained by fixing $d-1$ points on C and letting the d^{th} point move. In particular, $\gamma_j(C, \tilde{D}) \simeq C$. As before, all choices of C, \tilde{D} give a curve in the same linear equivalence class, and we will write γ_j for the linear equivalence class.

Definition 12. Fix a pencil of plane cubics \mathcal{P} and a base point a of the pencil. Let $\gamma_t(\mathcal{P}, a)$ be the family $(\mathcal{P}_t, d \cdot a)$.

As before, all choices give a curve in the same linear equivalence class, and we will write γ_t for the linear equivalence class.

γ_j and γ_t generate $H_2(\overline{\mathcal{Q}}_1(G(1, 1), d), \mathbb{Q})$. Their images under the induced map $i_{0*} : H_2(\overline{\mathcal{Q}}_1(G(1, 1), d), \mathbb{Q}) \rightarrow H_2(\overline{\mathcal{Q}}_1(\mathbb{P}^{n-1}, d), \mathbb{Q})$ are curves generating $H_2(\overline{\mathcal{Q}}_1(\mathbb{P}^{n-1}, d), \mathbb{Q})$. Let $\gamma_j^n = i_{0*}\gamma_j^1$ and $\gamma_t^n = i_{0*}\gamma_t^1$. When there is no source of confusion, we drop the subscripts. As the inclusion i_0 induces isomorphisms on both the homology and cohomology of $\overline{\mathcal{Q}}_1(G(1, 1), d)$ with $\overline{\mathcal{Q}}_1(\mathbb{P}^{n-1}, d)$, all intersection numbers computed in $\overline{\mathcal{Q}}_1(G(1, 1), d)$ are also valid in $\overline{\mathcal{Q}}_1(\mathbb{P}^{n-1}, d)$, so for simplicity we will always compute them in $\overline{\mathcal{Q}}_1(G(1, 1), d)$.

The intersection multiplicities between the divisors and curves defined on $\overline{\mathcal{Q}}_1(\mathbb{P}^{n-1}, d)$ are given in the following table.

	γ_j	γ_t
D_b	$d-1$	$-\frac{d(d-1)}{2}$
D_j	0	12
D_{fd}	$24(d-1)$	0

(14)

Proof. The intersections are straightforward to compute, with the exception of $\gamma_t \cdot D_b$, which we do here.

Recall that $\overline{\mathcal{Q}}_1(G(1, 1), d)$ is the quotient of $\overline{\mathcal{M}}_{1,0|d}$ by S_d , let $\varphi : \overline{\mathcal{M}}_{1,0|d} \rightarrow \overline{\mathcal{Q}}_1(G(1, 1), d)$ denote the quotient map. The preimage of D_b under this map is $\coprod_{i < j} \Delta_{ij}$, where Δ_{ij} is the divisor where $p_i = p_j$ in $\overline{\mathcal{M}}_{1,0|d}$.

We defined a pencil $\mathcal{P} \rightarrow B = \mathbb{P}^1$ with d sections, namely d times the section σ given by a base point a . By the universal property of $\overline{\mathcal{Q}}_1(G(1, 1), d)$, this defines a map $\mu : B \rightarrow \overline{\mathcal{Q}}_1(G(1, 1), d)$, and we defined the curve γ_t as the image $\mu(B)$. By definition

$$\gamma_t \cdot D_b = c_1(\mu^* \mathcal{N}_{D_b/\overline{\mathcal{Q}}_1(G(1,1),d)}).$$

By the universal property of $\overline{\mathcal{M}}_{1,0|d}$, μ in fact factors through $\overline{\mathcal{M}}_{1,0|d}$, and we have $\mu' : B \rightarrow \overline{\mathcal{M}}_{1,0|d}$. Let us write C_t for the image $\mu'(B)$ in $\overline{\mathcal{M}}_{1,0|d}$.

So

$$c_1(\mu^* \mathcal{N}_{D_b/\overline{\mathcal{Q}}_1(G(1,1),d)}) = c_1(\mu'^* \varphi^* \mathcal{N}_{D_b/\overline{\mathcal{Q}}_1(G(1,1),d)}) = c_1(\mu'^* \prod_{i < j} \mathcal{N}_{\Delta_{ij}/\overline{M}_{1,0|d}}) = \frac{d(d-1)}{2} c_1(\mu'^* \mathcal{N}_{\Delta_{12}/\overline{M}_{1,0|d}}).$$

It remains to compute the degree of $\mathcal{N}_{\Delta_{12}}$ on C_t . In the remainder, we denote Δ_{12} simply by Δ and $\overline{M}_{1,0|d}$ simply by M .

First, we reduce the problem to one involving only the vertical tangent space for the map $\overline{M}_{1,0|d} \rightarrow \overline{M}_{1,1}$, that is, we avoid deformations that change the modulus of the underlying curve.

$$\begin{array}{ccccccc}
& 0 & & 0 & & & \\
& \downarrow & & \downarrow & & & \\
0 & \longrightarrow & verT_{\Delta} & \longrightarrow & verT_M & \longrightarrow & \mathfrak{N}_{\Delta/M} \longrightarrow 0 \\
& \downarrow & & \downarrow & & & \\
0 & \longrightarrow & T_{\Delta} & \longrightarrow & T_M & \longrightarrow & \mathcal{N}_{\Delta/M} \longrightarrow 0 \\
& \downarrow & & \downarrow & & & \\
0 & \longrightarrow & T_{\overline{M}_{1,1}} & \longrightarrow & T_{\overline{M}_{1,1}} & \longrightarrow & 0 \longrightarrow 0 \\
& \downarrow & & \downarrow & & & \\
& 0 & & 0 & & &
\end{array}$$

By the snake lemma, $\mathcal{N}_{\Delta/M} = \mathfrak{N}_{\Delta/M}$. To compute $\mathfrak{N}_{\Delta/M}$, we use the fact that the sum of the tangent spaces at each marked point surjects onto $verT_{\Delta}$.

$$\begin{array}{ccccccc}
& 0 & & 0 & & & \\
& \downarrow & & \downarrow & & & \\
0 & \longrightarrow & K_1 & \xrightarrow{\simeq} & K_2 & \longrightarrow & 0 \longrightarrow 0 \\
& \downarrow & & \downarrow & & & \\
0 & \longrightarrow & \oplus_{i=2}^d T_{p_i} & \xrightarrow{\Delta \oplus id^{d-2}} & \oplus_{i=2}^d T_{p_i} & \longrightarrow & Q \longrightarrow 0 \\
& \downarrow & & \downarrow & & & \\
0 & \longrightarrow & verT_{\Delta} & \longrightarrow & verT_M & \longrightarrow & \mathfrak{N}_{\Delta/M} \longrightarrow 0 \\
& \downarrow & & \downarrow & & & \\
& 0 & & 0 & & &
\end{array}$$

Again by the snake lemma, $\mathfrak{N}_{\Delta/M}$ is isomorphic to the quotient Q . Finally we can compute

$$\mathcal{N}_{\Delta/M}|_{C_t} = Q|_{C_t} = \bigoplus_{i=1}^d T_{p_i}|_{C_t} - \bigoplus_{i=2}^d T_{p_i}|_{C_t} = T_{p_1}|_{C_t} = -\psi_1|_{C_t} = -1,$$

where ψ_1 is the standard cotangent class to p_1 .

We conclude $c_1(\mu'^* \mathcal{N}_{\Delta_{12}/\overline{M}_{1,0|d}}) = -1$ and

$$c_1(\mu^* \mathcal{N}_{D_b/\overline{Q}_1(G(1,1),d)}) = c_1(\mu'^* \prod_{i < j} \mathcal{N}_{\Delta_{ij}/\overline{M}_{1,0|d}}) = -\frac{d(d-1)}{2},$$

as desired. \square

We deduce from Table 14 that

Lemma 4.2. $D_{fd} = d(d-1)D_j + 24D_b$.

4.3 Nef Cone

Recall that on a variety X , a divisor D is called nef if $D \cdot C = 0$ for every irreducible curve $C \subset X$. To show D is nef, it is sufficient to prove that it is base point free, because given any curve C and a point $x \in C$, if D is base point free, there is D' numerically equivalent to D such that D' does not contain x , so it does not contain C , so $D \cdot C = D' \cdot C \geq 0$.

Theorem 4.3. *The nef cone of $\overline{Q}_1(\mathbb{P}^{n-1}, d)$ is bounded by the divisors D_j and D_{fd} .*

Proof. We begin by showing D_j is base point free, hence nef. Take any point $x = (C, [s_0, \dots, s_{d-1}]) \in \overline{Q}_1(\mathbb{P}^{n-1}, d)$. Fix ρ not equal to the j -invariant of C . Then $D_j(\rho)$ is a representative of the class D_j which does not contain x .

As computed in Table 14, $D_j \cdot \gamma_j = 0$, and γ_j is an effective curve. As D_j is nef and has zero intersection with an effective curve, it is on the boundary of the nef cone.

Now we show that D_{fd} is also base point free. Again take any point $x = (C, [s_0, \dots, s_{d-1}]) \in \overline{Q}_1(\mathbb{P}^{n-1}, d)$. At least one $s_i \neq 0$. Let H be the hyperplane $x_i = 0$ in \mathbb{P}^{n-1} . Let $y := (C, |s_H|) \in \overline{Q}_1(G(1,1), d)$. Using again the maps from diagram 4.1, $\pi(\varphi^{-1}(y))$ is a finite set in $\overline{M}_{1,0|2}$. By a dimension count, we can choose a pencil of plane cubics \mathcal{P} and two basepoints a, b of \mathcal{P} such that the associated curve $B_{\mathcal{P}_{a,b}} \subset \overline{M}_{1,0|2}$ does not meet $\pi(\varphi^{-1}(y))$. For such a choice of $\mathcal{P}_{a,b}$, $D_{fd}(H, \mathcal{P}_{a,b})$ is a representative of the class D_{fd} which does not contain the point x . Hence D_{fd} is base point free, hence nef.

Again, it was computed in Table 14, that $D_{fd} \cdot \gamma_t = 0$. As D_{fd} is nef and has zero intersection with an effective curve, it is on the boundary of the nef cone. \square

4.4 Projectivity

The above analysis of the cone of nef divisors and its dual, the cone of effective curves, yields a simple proof of the projectivity of $\overline{Q}_1(\mathbb{P}^{n-1}, d)$.

Corollary 4.4. $\overline{Q}_1(\mathbb{P}^{n-1}, d)$ is a projective scheme.

Proof. If $d = 1$, $\overline{Q}_1(\mathbb{P}^{n-1}, 1) \simeq \mathbb{P}^{n-1} \times \overline{M}_{1,1}$ [MOP09] Section 2.5. Hence for $d = 1$ $\overline{Q}_1(\mathbb{P}^{n-1}, d)$ is projective.

Now assume $d > 1$. Let $D = D_j + D_{fd}$. We show D is ample, by checking positivity on the closure of the cone of effective curves. As γ_j and γ_t span the cone of effective curves, any curve class in the closure of the cone of effective curves can be represented as $E_{a,b} = a\gamma_j + b\gamma_t$ for a, b each nonnegative and not both zero. By the computation in Table 14, $D \cdot E_{a,b} = 24(d-1)a + 12b$, which is strictly positive by our assumptions on (a, b) and the assumption $d > 1$. We conclude D is ample, and hence $\overline{Q}_1(\mathbb{P}^{n-1}, d)$ is projective for all d . \square

4.5 Effective Cone

Boucksom, Demailly, Paun and Peternell showed that for a projective variety (more generally, compact Kähler manifold) the cone of pseudo-effective divisors is dual to the cone of moving curves [BDPP04]. For a variety X , a moving curve is a linear equivalence class of curves $[\gamma]$ such that for any point $x \in X$, there is a representative γ of $[\gamma]$ such that $x \in \gamma$.

Theorem 4.5. The effective cone of $\overline{Q}_1(\mathbb{P}^{n-1}, d)$ is bounded by the divisors D_j and D_b .

Proof. By construction, the divisors D_j and D_b are effective. To show that in fact they lie on the boundary, we demonstrate for each a moving curve that has intersection number 0 with the divisor. Note that to show a curve class $[\gamma]$ is moving, it suffices to show that through any *general* point $y \in X$ there is a representative γ such that $y \in \gamma$. This is because for any $x \in X$, we can take a 1-parameter family of general points $y_t \rightarrow x$ as $t \rightarrow 0$. If for all $t \neq 0$ there is a curve γ_t in the class $[\gamma]$ through y_t , then the limit of these curves γ_0 contains $x = y_0$ and is also in the class $[\gamma]$.

For D_j , the curve γ_j serves this purpose. In Section 4.2, we saw that $\gamma_j \cdot D_j = 0$. It remains to see that γ_j is a moving curve. Take a general point $(C, D) \in \overline{Q}_1(\mathbb{P}^{n-1}, d)$, and by generality assume C is smooth. Throw away a point of D and let \tilde{D} be the resulting divisor of degree $d-1$. The representative $\gamma_j(C, \tilde{D})$ contains (C, D) , so γ_j is a moving curve, as desired.

For the divisor D_b , we demonstrate the existence of a moving curve which does not intersect it. In the Hilbert scheme of degree $m+1$ curves ξ in \mathbb{P}^m , let X be the open subscheme where the curve is smooth. X has dimension $(m+1)^2$. For any point $z \in \mathbb{P}^m$, let X_z be the subscheme of X of curves ξ passing through z . X_z has codimension $m-1$ in X . Take d general points z_1, \dots, z_d , and let $X_{z_1, \dots, z_d} = X_{z_1} \cap \dots \cap X_{z_d}$. If we take $m \geq d-3$, X_{z_1, \dots, z_d} will have dimension ≥ 4 , in particular, it will be nonempty.

Take a curve $\gamma \subset X_{z_1, \dots, z_d}$. Restricting the universal family of the Hilbert scheme to γ gives a 1 dimensional family of smooth genus 1 curves in \mathbb{P}^m , each with d *distinct* marked points z_1, \dots, z_d . By the universal property of $\overline{Q}_1(G(1, 1), d)$, this maps to a curve in $\overline{Q}_1(G(1, 1), d)$, and let $\gamma_{nt} \subset \overline{Q}_1(G(1, 1), d)$ be its closure.

For all $(C, D) \in \gamma_{nt}$, D is an effective divisor of d distinct points, so $\gamma_{nt} \cdot D_b = 0$. It remains to see that γ_{nt} is a moving curve. Fix any general point $(C, D) \in \overline{Q}_1(G(1, 1), d)$. By genericity, C is smooth and D consists of distinct points. Pick any embedding of $i : C \hookrightarrow \mathbb{P}^m$, and label the points of $i(D)$ as z_1, \dots, z_d . Now the image γ_{nt} of any $\gamma \subset X_{z_1, \dots, z_d}$ contains (C, D) , as desired. We conclude that D_b is a boundary of the effective cone of $\overline{Q}_1(\mathbb{P}^{n-1}, d)$. \square

4.6 Canonical Divisor

Theorem 4.6. $K_{\overline{Q}_1(\mathbb{P}^{n-1}, d)} = \frac{(d-11+(d-1)(n-1))}{12} D_j - n D_b$.

Proof. We begin with the $n = 1$ case.

The map $\varphi : \overline{M}_{1,0|d} \xrightarrow{/S_d} \overline{Q}_1(G(1, 1), d)$ is ramified exactly along the divisor $\sum_{i < j} \Delta_{ij}$. We can use the Riemann-Hurwitz formula to deduce $K_{\overline{Q}_1(G(1,1),d)}$ from the canonical divisor of $\overline{M}_{1,0|d}$,

$$K_{\overline{M}_{1,0|d}} = 13\lambda - 2\Delta_{nodal} + \psi_{tot},$$

which was computed by Hassett [Ha03]. Here Δ_{nodal} is the locus where the genus 1 curve is nodal, $\psi_{tot} = \sum_i \psi_i$ and λ, ψ_i are the standard Hodge and cotangent classes.

By Riemann-Hurwitz,

$$\varphi^* K_{\overline{Q}_1(G(1,1),d)} = K_{\overline{M}_{1,0|d}} - R = 13\lambda - 2\Delta_{nodal} + \psi_{tot} - \sum_{i < j} \Delta_{ij} \quad (15)$$

Writing $K_{\overline{Q}_1(G(1,1),d)} = \alpha D_j + \beta D_b$, $\varphi^* K_{\overline{Q}_1(G(1,1),d)} = 12\alpha\lambda + \beta \sum_{i < j} \Delta_{ij}$ and

$$12\alpha\lambda + \beta \sum_{i < j} \Delta_{ij} = (13 - 24 + d)\lambda - \sum_{i < j} \Delta_{ij}$$

so

$$K_{\overline{Q}_1(G(1,1),d)} = \frac{d-11}{12} D_j - D_b. \quad (16)$$

For $n > 1$, we use the inclusion $i_0 : \overline{Q}_1(G(1, 1), d) \hookrightarrow \overline{Q}_1(\mathbb{P}^{n-1}, d)$, which gives the short exact sequence

$$0 \rightarrow T_{\overline{\mathbb{Q}}_1(G(1,1),d)} \rightarrow i_0^* T_{\overline{\mathbb{Q}}_1(\mathbb{P}^{n-1},d)} \rightarrow \mathcal{N}_{\overline{\mathbb{Q}}_1(G(1,1),d)/\overline{\mathbb{Q}}_1(\mathbb{P}^{n-1},d)} \rightarrow 0. \quad (17)$$

Take a family $\pi \downarrow \uparrow \sigma_1 + \dots + \sigma_d$
 \mathcal{C}
 B

over a curve $B \subset i_0(\overline{\mathbb{Q}}_1(G(1,1),d))$. Then the normal bundle is

$$\mathcal{N}_{\overline{\mathbb{Q}}_1(G(1,1),d)/\overline{\mathbb{Q}}_1(\mathbb{P}^{n-1},d)}|_B = \pi_* \left(\mathcal{O}(\sum \sigma_i)^{n-1} \right).$$

We can use Grothendieck-Riemann-Roch to compute the first chern class of this bundle.

$$\begin{aligned} ch \left(\pi_* \mathcal{O}(\sum \sigma_i)^{n-1} \right) &= \pi_* \left(ch \left(\mathcal{O}(\sum \sigma_i)^{n-1} \right) td(T_{\mathcal{C}/B}) \right) \\ &= \pi_* \left[\left((n-1) + (n-1)(\sum \sigma_i) + \frac{(n-1)^2 - 2 \binom{n-1}{2}}{2} (\sum \sigma_i)^2 \right) \left(1 - \frac{1}{2}\gamma + \frac{1}{12}(\gamma^2 + \eta) + \dots \right) \right] \\ &= \pi_* \left[(n-1) \left(1 + (\sum \sigma_i) + \frac{1}{2}(\sum \sigma_i)^2 \right) \left(1 - \frac{1}{2}\gamma + \frac{1}{12}(\gamma^2 + \eta) + \dots \right) \right] \end{aligned}$$

Taking the degree 1 part of each side,

$$\begin{aligned} c_1 \left(\pi_* \mathcal{O}(\sum \sigma_i)^{n-1} \right) &= \pi_* \left[(n-1) \left(\frac{1}{12}(\gamma^2 + \eta) - \frac{1}{2}\gamma \cdot (\sum \sigma_i) + \frac{1}{2}(\sum \sigma_i)^2 \right) \right] \\ &= (n-1) \left(\lambda - \frac{\pi_*(\gamma \cdot \sum \sigma_i)}{2} + \frac{\pi_*(\sum \sigma_i)^2}{2} + \frac{\pi_*(\sum_{i \neq j} \sigma_i \cdot \sigma_j)}{2} \right) \end{aligned}$$

$\mathcal{O}(\sigma_i^2)$ is the normal bundle to σ_i , so

$$\pi_*(\sigma_i^2) = -\psi_i.$$

Meanwhile

$$\gamma \cdot \sigma_i = c_1(\omega_{\mathcal{C}/B}|_{\sigma_i}) = \psi_i.$$

So

$$c_1 \left(\pi^* \mathcal{N}_{\overline{\mathbb{Q}}_1(G(1,1),d)/\overline{\mathbb{Q}}_1(\mathbb{P}^{n-1},d)} \right) = (n-1) \left(\lambda - \psi_{tot} + \sum_{i < j} \Delta_{ij} \right).$$

Writing

$$c_1 \left(\mathcal{N}_{\overline{\mathbb{Q}}_1(G(1,1),d)/\overline{\mathbb{Q}}_1(\mathbb{P}^{n-1},d)} \right) = \alpha D_j + \beta D_b,$$

$$12\alpha\lambda + \beta \sum_{i < j} \Delta_{ij} = (n-1)((1-d)\lambda + \sum_{i < j} \Delta_{ij})$$

and we conclude

$$c_1 \left(N_{\overline{\mathbb{Q}}_1(G(1,1),d)/\overline{\mathbb{Q}}_1(\mathbb{P}^{n-1},d)} \right) = \frac{(n-1)(1-d)}{12} D_j + (n-1) D_b.$$

Taking the first chern class of each term of (17) and using the expression 16 for $K_{\overline{\mathbb{Q}}_1(G(1,1),d)}$, we conclude

$$\begin{aligned} c_1(i_0^* T_{\overline{\mathbb{Q}}_1(G(1,1),d)}) &= c_1(T_{\overline{\mathbb{Q}}_1(G(1,1),d)}) + c_1(\mathcal{N}_{\overline{\mathbb{Q}}_1(G(1,1),d)/\overline{\mathbb{Q}}_1(\mathbb{P}^{n-1},d)}) \\ &= - \left((d-11) \frac{D_j}{12} - D_b \right) + (n-1) \left((1-d) \frac{D_j}{12} + D_b \right) \end{aligned}$$

Hence

$$i_0^* K_{\overline{\mathbb{Q}}_1(G(1,1),d)} = \frac{(d-11 + (d-1)(n-1))}{12} D_j - n D_b.$$

Because i_0^* is an isomorphism of the Picard groups of $\overline{\mathbb{Q}}_1(G(1,1),d)$ and $\overline{\mathbb{Q}}_1(\mathbb{P}^{n-1},d)$, we conclude that for all $n \geq 1$,

$$K_{\overline{\mathbb{Q}}_1(\mathbb{P}^{n-1},d)} = \frac{(d-11 + (d-1)(n-1))}{12} D_j - n D_b,$$

as desired. □

Proposition 4.7. $\overline{\mathbb{Q}}_1(\mathbb{P}^{n-1}, d)$ is Fano iff $n(d+2)(d-1) < 20$.

Proof. Recall that a scheme X is Fano if $-K_X$ ample. We use Lemma 4.2 to write $-K_{\overline{\mathbb{Q}}_1(\mathbb{P}^{n-1},d)}$ in terms of the divisors bounding the nef cone. We find

$$-K_{\overline{\mathbb{Q}}_1(\mathbb{P}^{n-1},d)} = \frac{1}{24} (n D_{fd} - (n(d+2)(d-1) - 20) D_j).$$

□

5 Rational connectedness

Recall that a scheme X is *rationally connected* if for any two general points $x_1, x_2 \in X$, there is a chain of rational curves R_1, \dots, R_k such that $x_1 \in R_1$ and $x_2 \in R_k$.

Theorem 5.1. $\overline{Q}_1(\mathbb{P}^{n-1}, d)$ is rationally connected.

Proof. First, consider the case of $\overline{Q}_1(G(1, 1), d)$. As we need show only that two general points are connected by a chain of rational curves, it suffices to fix two general points $(C, D), (C', D')$ in \tilde{U}_d . By Proposition 2.8, $\tilde{U}_d \rightarrow \tilde{M}_{1,1}$ is a fiber bundle with fibers \mathbb{P}^{d-1}/\hat{H} . So the fibers are rationally connected. (To construct a line in \mathbb{P}^{d-1}/\hat{H} between any two points p_1 and p_2 , take a cover $\pi : \mathbb{P}^{d-1} \rightarrow \mathbb{P}^{d-1}/\hat{H}$, and two preimages q_1, q_2 of p_1, p_2 . There is a line $L \subset \mathbb{P}^{d-1}$ between q_1 and q_2 . The quotient $\pi(L)$ is rational because any quotient of \mathbb{P}^1 by a finite group is isomorphic to \mathbb{P}^1 , and so $\pi(L)$ is a rational curve in \mathbb{P}^{d-1}/\hat{H} connecting p_1 and p_2 .)

Let R_1 be a rational curve between (C, D) and (C, dp) , where dp is the divisor d times a point p . Let R_2 be a rational curve between (C, dp) and (C', dp') (a copy of $\overline{M}_{1,1}$ in $\overline{Q}_1(G(1, 1), d)$). And let R_3 be a rational curve between (C', dp') and (C', D') . This chain of rational curves connects the two general points we started with.

In the general case $n > 1$, again fix two general stable quotients $(C, [s_0, \dots, s_{n-1}])$ and $(C', [s'_0, \dots, s'_{n-1}])$ whose underlying curve is a smooth genus 1 curve other than ξ_4 or ξ_6 . Let R_1 be the closure of $(C, [s_0, \epsilon s_1, \dots, \epsilon s_{n-1}])$, $\epsilon \in \mathbb{C}$, and R_5 the closure of $(C', [s'_0, \epsilon s'_1, \dots, \epsilon s'_{n-1}])$, $\epsilon \in \mathbb{C}$. By genericity, s_0 and s'_0 are both nonzero.

Recall the inclusion $i_0 : \overline{Q}_1(G(1, 1), d) \rightarrow \overline{Q}_1(\mathbb{P}^{n-1}, d)$ (Equation 4). R_1 and R_5 both intersect the image $i_0(\overline{Q}_1(G(1, 1), d))$, namely at $(C, |s_0|)$ and $(C', |s'_0|)$, where $|s|$ denotes the zero set of the section. As discussed above, there is a chain of rational curves R_2, R_3, R_4 in $i_0(\overline{Q}_1(G(1, 1), d))$ connecting those two points. Taken together, R_1, \dots, R_5 is then a chain of rational curves connecting the original two general points, as desired. \square

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